Numerical Solution of Differential Equations II. QS 4

Question 1.

Consider the linear advection problem

$$u_t + au_x = 0. \tag{1}$$

- Recall the concepts of consistency, stability, and convergence. Comment on their importance, and how they relate to each other.
- The forward Euler in time, centred space finite difference approximation to (1) gives an unstable method. One could also approximate the time derivative by a centred scheme (so-called Leapfrog method)

$$u_i^{n+1} = u_i^{n-1} - \frac{a}{2}\lambda(u_{i+1}^n - u_{i-1}^n), \tag{2}$$

  with $\lambda = k/h$.
  - Draw the stencil of (2).
  - Knowing that (2) is second order accurate in space and time (you can also prove it, if feeling motivated), comment the drawbacks of the Leapfrog scheme with respect to Lax-Friedrichs or Lax-Wendroff methods.
- Show that the Lax-Friedrichs method applied to (1) is stable provided that $|a|\Delta t/h \leq 1$. **Notice that there was a typo in the lecture notes, on the definition of Lax-Friedrichs. It should read**

$$u_j^{n+1} = \frac{1}{2}(u_{j+1}^n + u_{j-1}^n) - \frac{ak}{2h}(u_{j+1}^n - u_{j-1}^n).$$

Question 2.

Take $a = 1$ and let u be the solution to (1) in $(-1, 1)$ with the initial condition

$$u(x, 0) = u_0(x) = \begin{cases} 1 & x < 0, \\ 0 & x > 0 \end{cases} \tag{3}$$

and the boundary conditions

$$u(-1, t) = 1, \quad u(1, t) = 0. \tag{4}$$

The exact solution is $u_0(x - at)$. Consider also the "Beam-Warming" scheme

$$u_i^{n+1} = u_i^n - \frac{a}{2}\lambda(3u_i^n - 4u_{i-1}^n + u_{i-2}^n) + \frac{a^2}{2}\lambda^2(u_i^n - 2u_{i-1}^n + u_{i-2}^n).$$

For each of the following schemes: Lax-Friedrichs, Lax-Wendroff, Beam-Warming

- Implement the scheme (in e.g. Matlab) to solve (1)-(3)-(4) in the space domain $(-1, 1)$ over the time interval $[0, 1/2]$, using $h = 0.0025$ and $\lambda = 1/2$.
- Visualize both numerical and exact solutions obtained with each scheme and comment on the resulting plots.
- Generate the error history of the numerical approximation: plot in log-log the error (choose some adequate norm) as function of the resolution h, while keeping the ratio $\lambda = 1/2$.
- What is the "experimental" convergence rate (the one suggested by these plots)?
- What is the "expected" accuracy (which would hold for smooth solutions)?