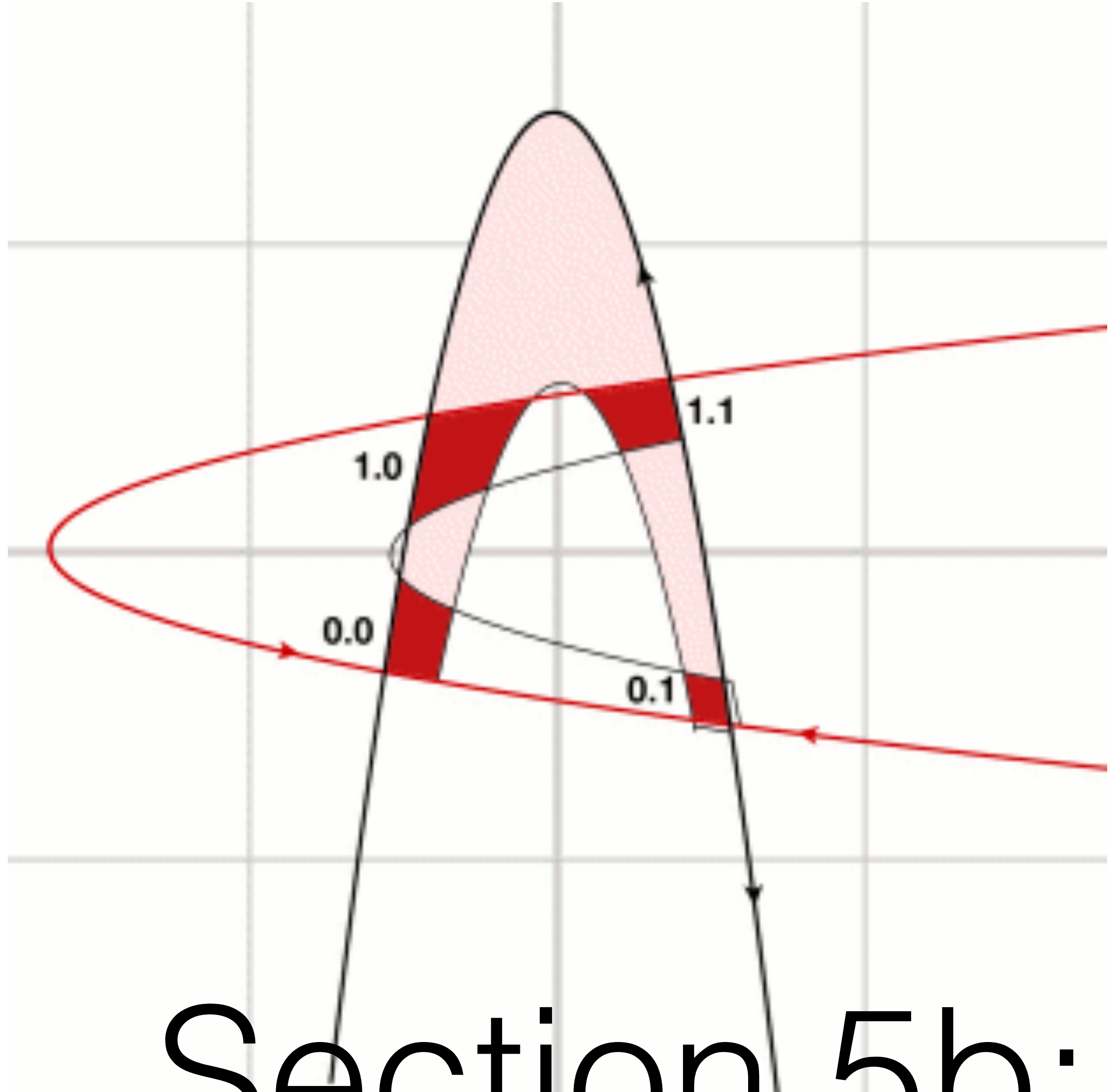



# Section 5b:
# Homoclinic Chaos

# 1. Bernoulli shift

# Dynamical systems

1. The space on which it is defined
2. The dynamics mapping sets to sets

Ex: Bernoulli shift

1. The space of bi-infinite sequence: $\Sigma$
2. a shift mapping sequences to sequences

$$\sigma: \Sigma \rightarrow \Sigma$$

Analysis of this system reveals infinitely periodic orbits infinitely quasi-periodic orbits, a dense orbit.

=>Sensitive dependence to i.c. and Chaos

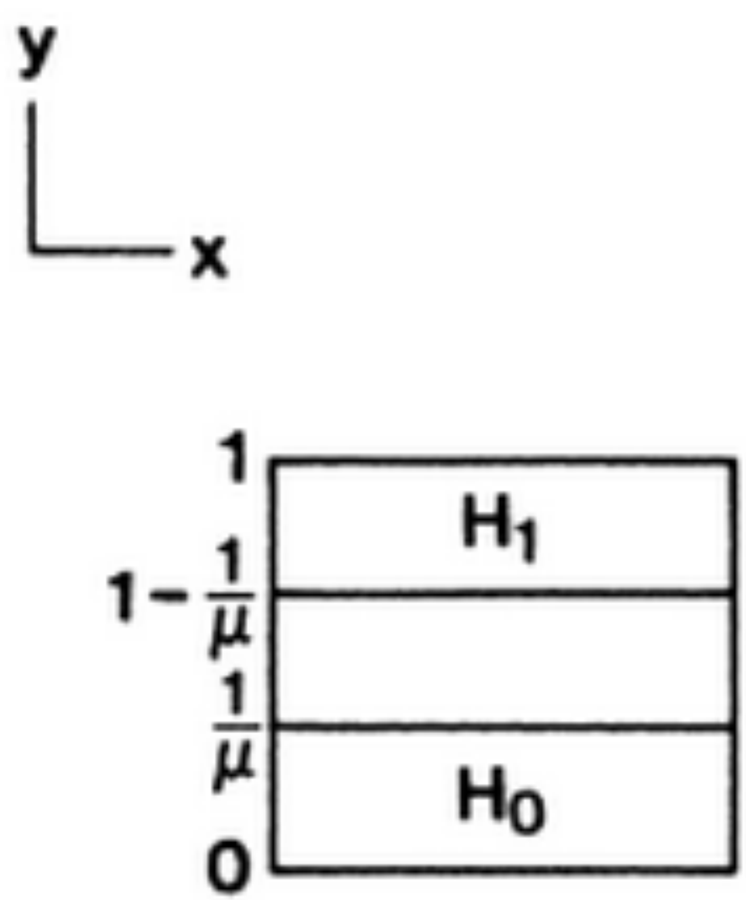
# 2. SMALE’S HORSESOE

# SMALE's HORSESHOE

First, define two rectangular regions in the unit square

$$H_0 = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1,\ 0 \leq y \leq 1/\mu\}$$

$$H_1 = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1,\ 1 - 1/\mu \leq y \leq 1\}$$

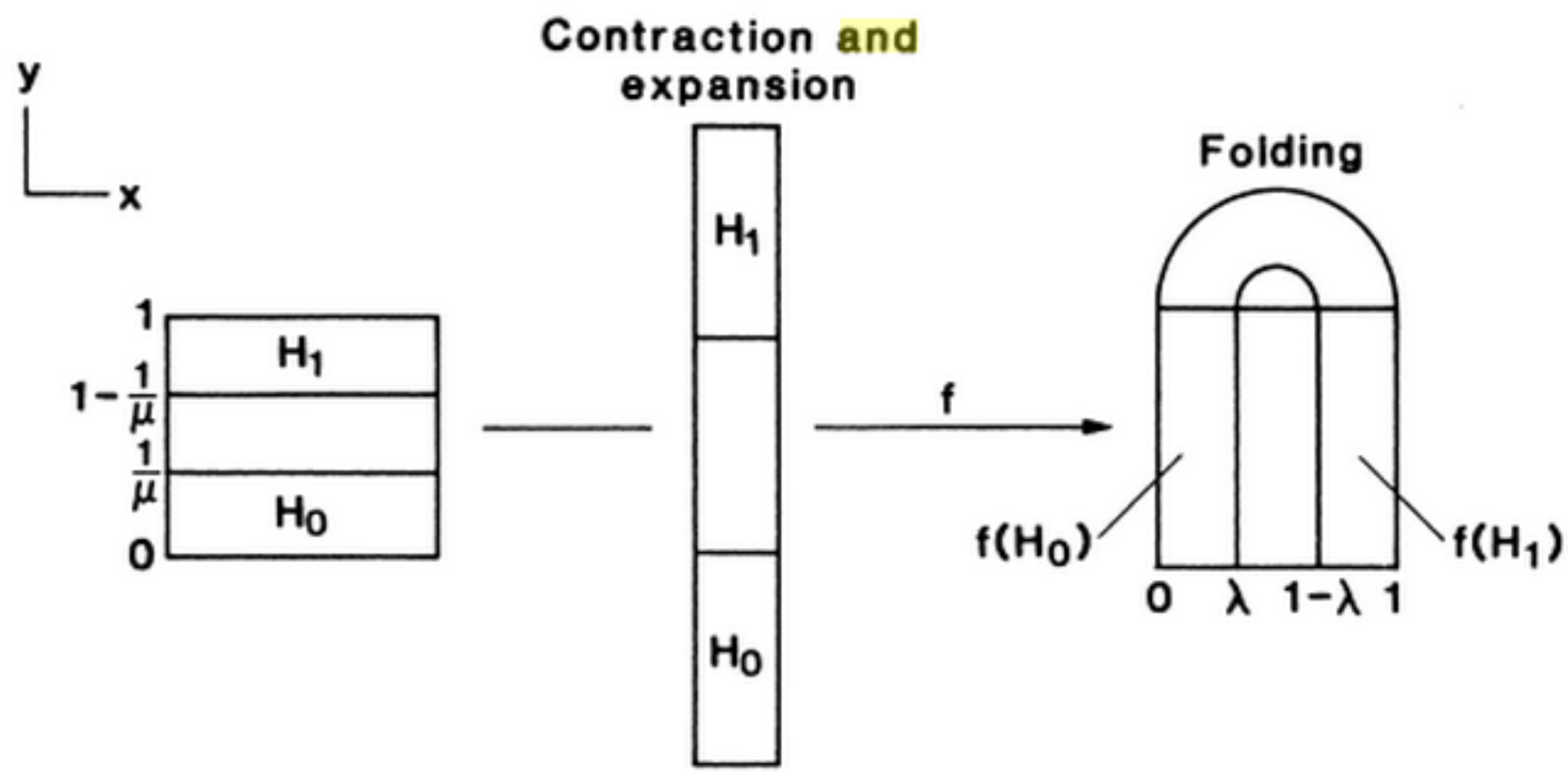


# SMALE's HORSESHOE

Second, define a map of these rectangle into themselves

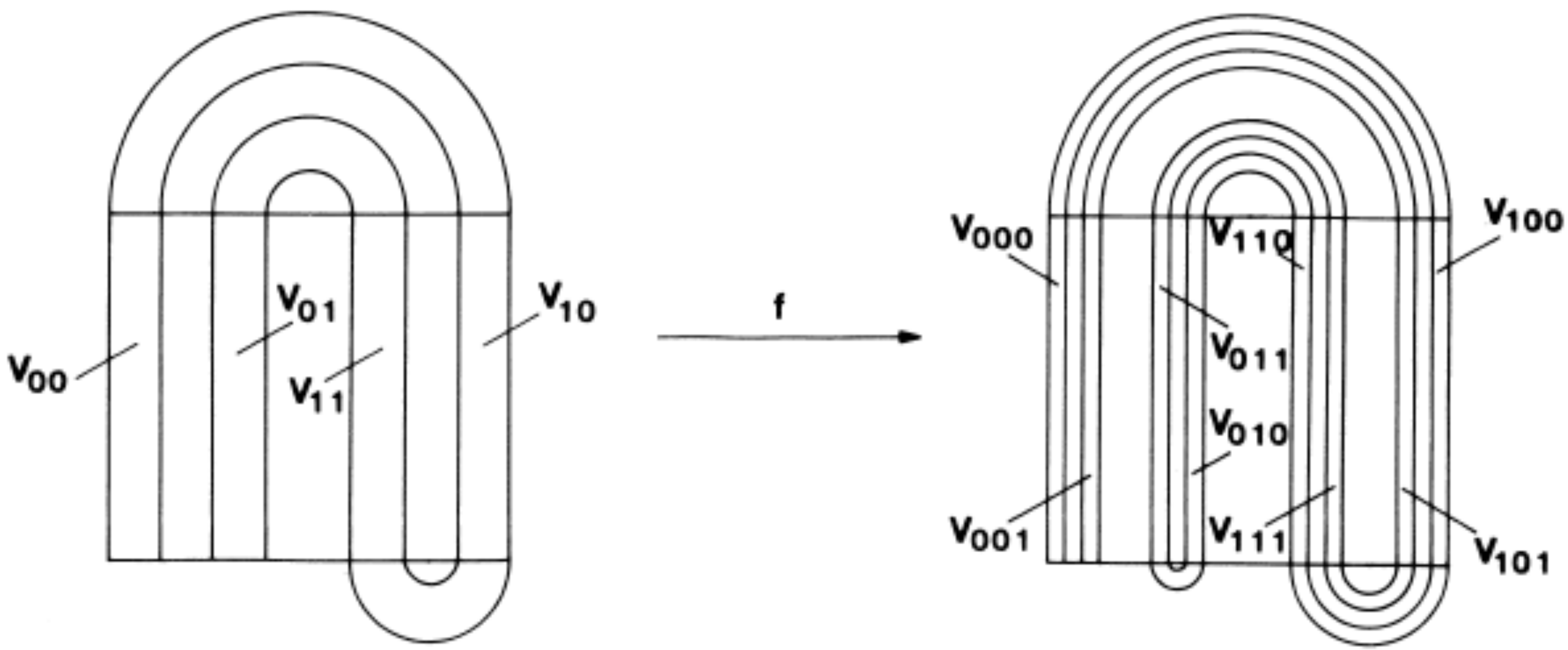
$$H_0 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$H_1 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -\lambda & 0 \\ 0 & -\mu \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ \mu \end{pmatrix}$$



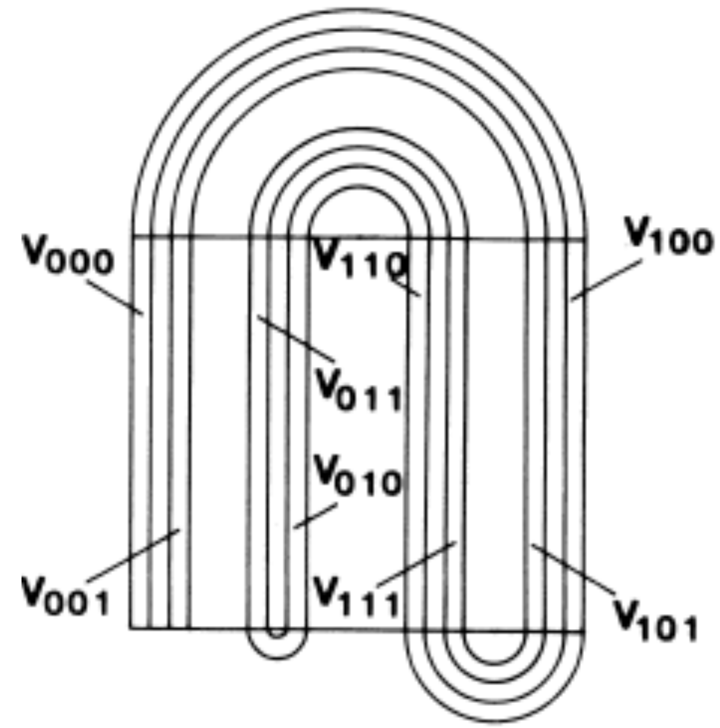
# SMALE's HORSESHOE

Third, repeat the operation

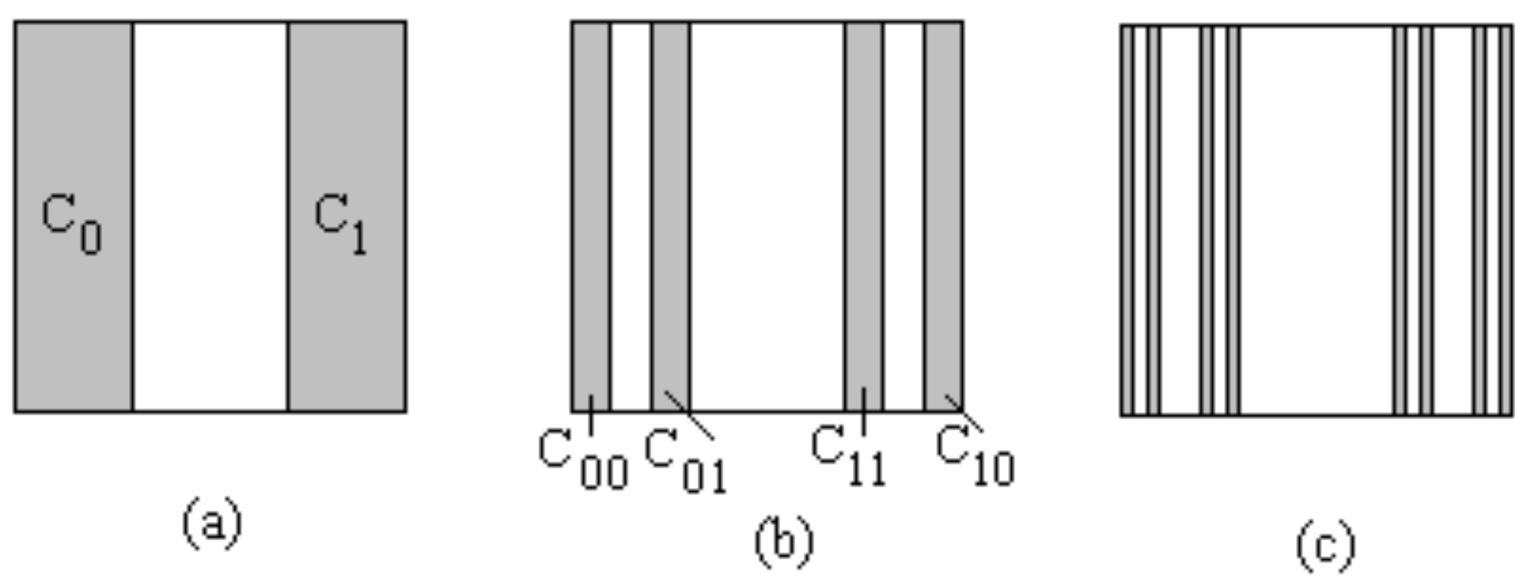


# SMALE's HORSESHOE

Fourth, introduce a coding

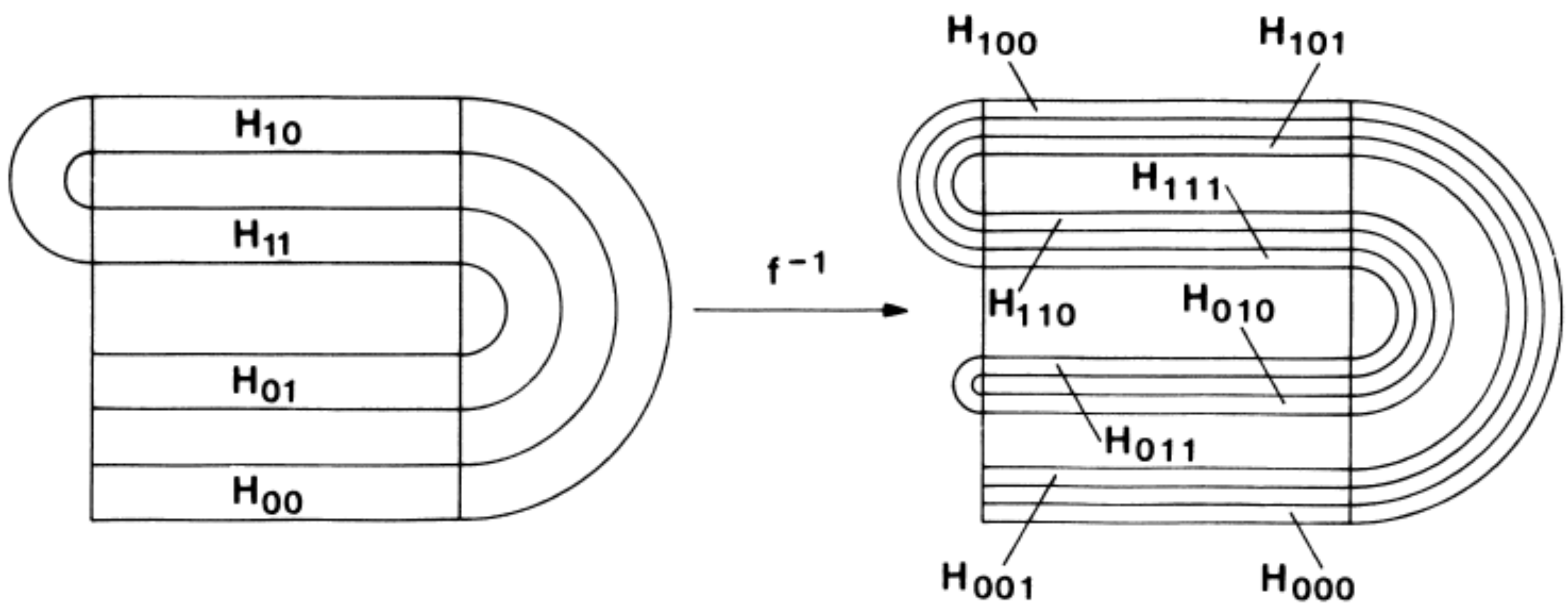



Coding: 01 means
that it was right at the first iteration left at the second

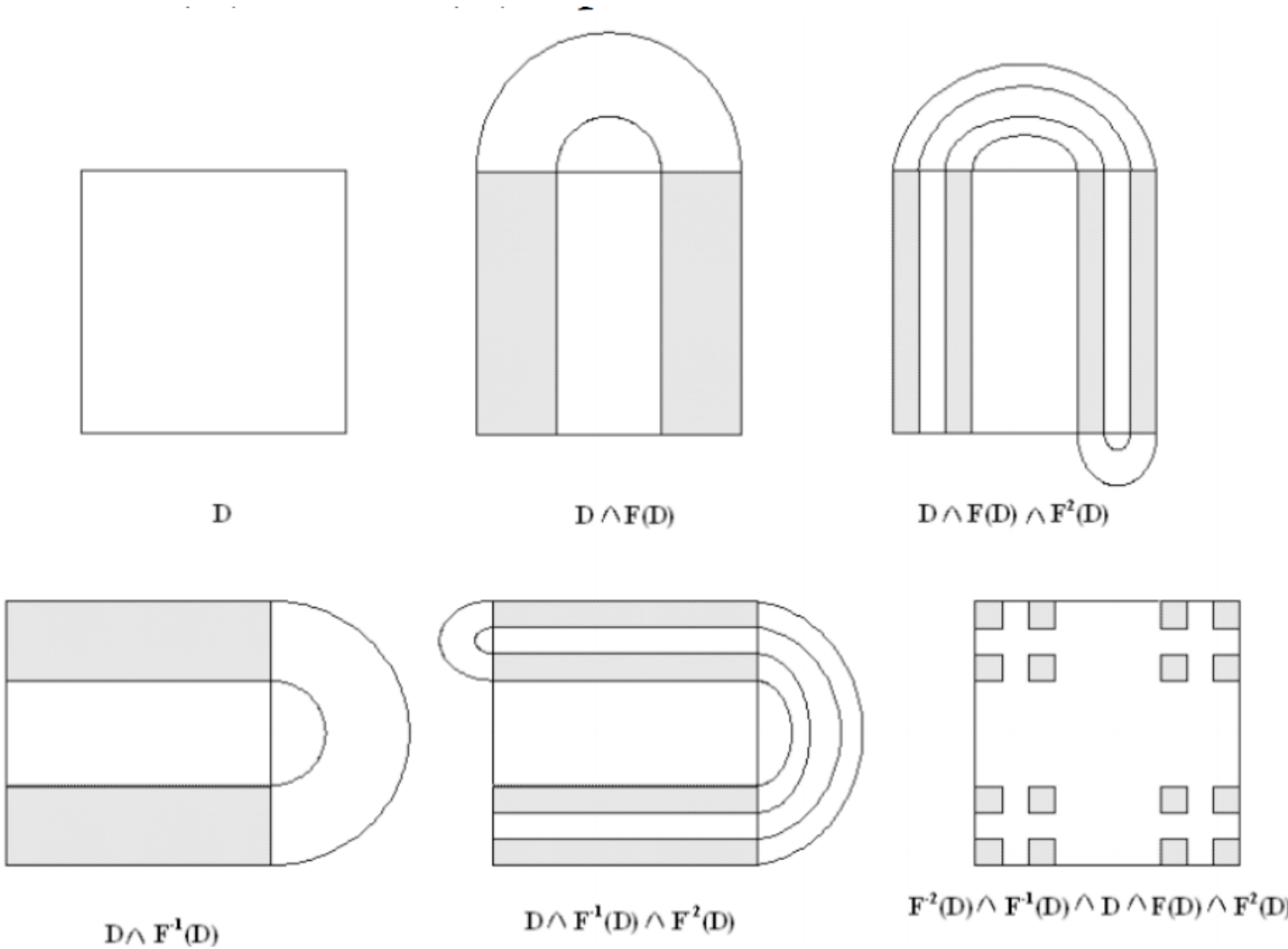


# SMALE's HORSESHOE

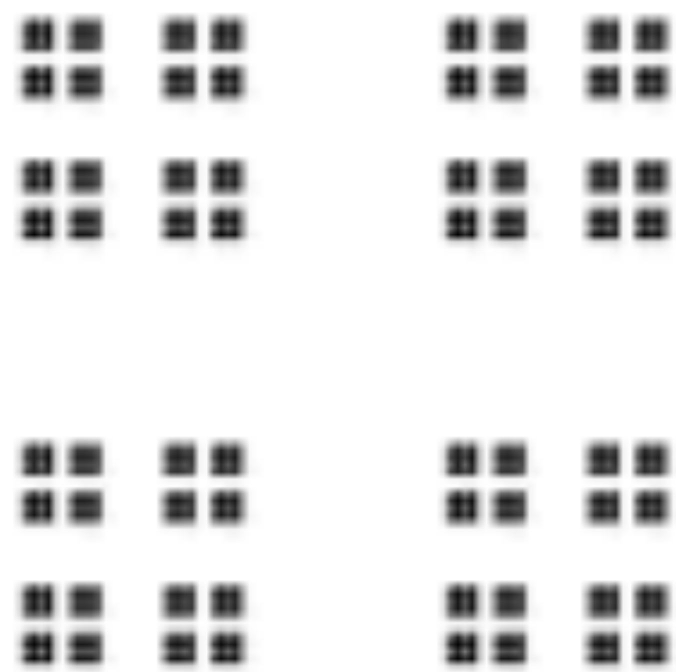
Fifth, do the same operation for the inverse map



# Sixth, take the intersection between the two sets



# SMALE's HORSESHOE



$$\Lambda = \bigcap_{n \in \mathbb{Z}^+} H^n \cap \bigcap_{n \in \mathbb{Z}^+} V^n = \bigcap_{n \in \mathbb{Z}} f^n(H_0 \cup H_1).$$

This set is a the intersection of a Cantor set of vertical lines and a cantor sets of horizontal lines

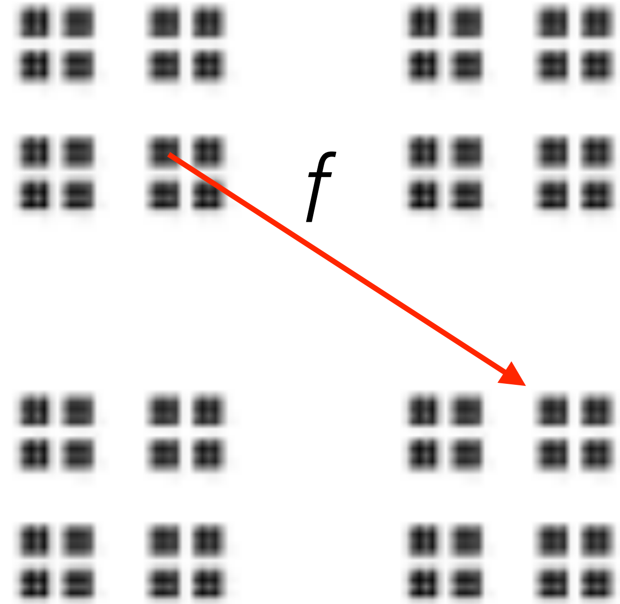
$$\Lambda = \bigcap_{n\in\mathbb{Z}^+} H^n \cap \bigcap_{n\in\mathbb{Z}^+} V^n = \bigcap_{n\in\mathbb{Z}} f^n(H_0 \cup H_1).$$

Each point in $\Lambda$ can be coded by two binary sequences. The first sequence codes its vertical position, the second sequence codes its horizontal position

Therefore to each point *p* in $\Lambda$ we can associate a bi-infinite sequence σ in *Σ*. Let

$$h: \Lambda \rightarrow \Sigma$$

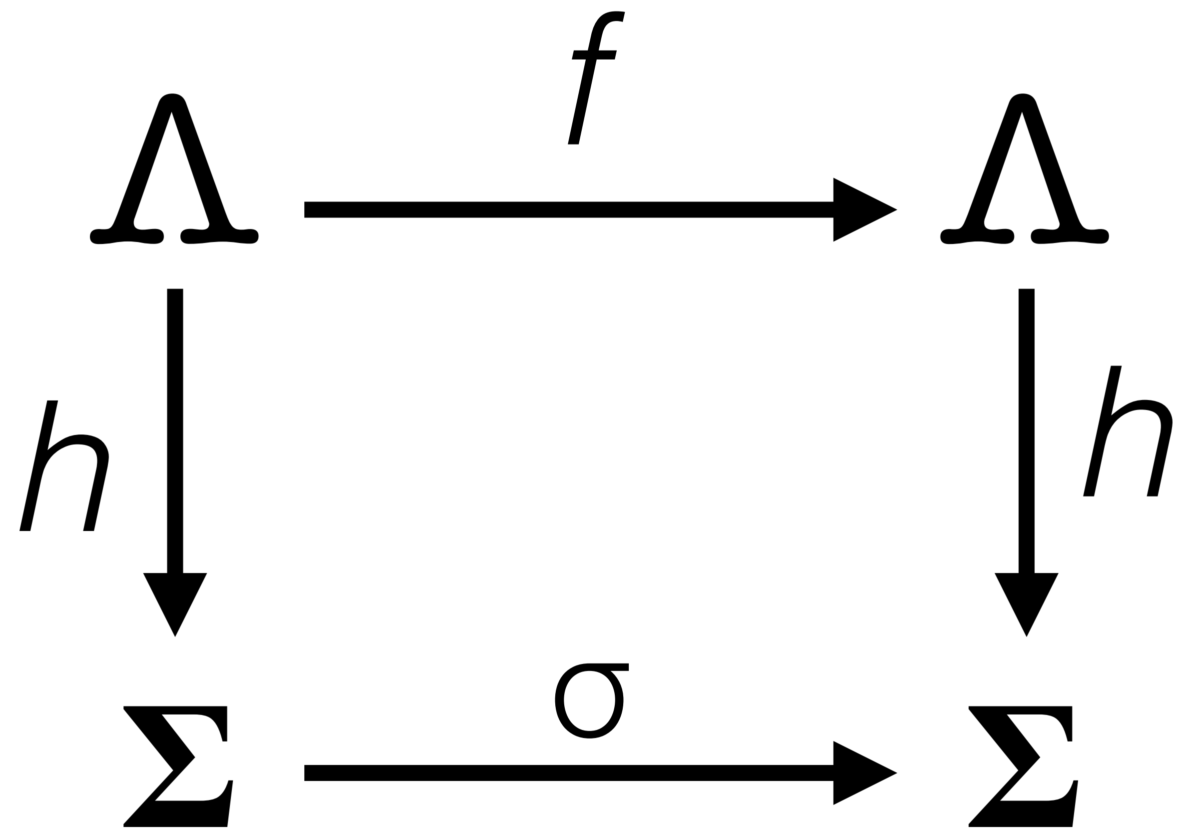
This map *h* is a homeomorphism (1:1, onto, continuous with continuous inverse).




The dynamics on $\Lambda$ is given by $f: \Lambda \rightarrow \Lambda$.
It maps points to points.
By construction, $\Lambda$ is an invariant set

Since there exists a homeomorphism *h*, it implies that the dynamics of *f* on $\Lambda$ is t*opologically conjugate* to the dynamics of σ on $\Sigma$.

# Topological equivalence




To each orbit in $\Sigma$ there is a corresponding orbit on $\Lambda$

Therefore, the system has, a countable infinity of periodic orbits, an uncountable infinity of non-periodic orbit, a dense orbit, sensitivity dependence to initial conditions.

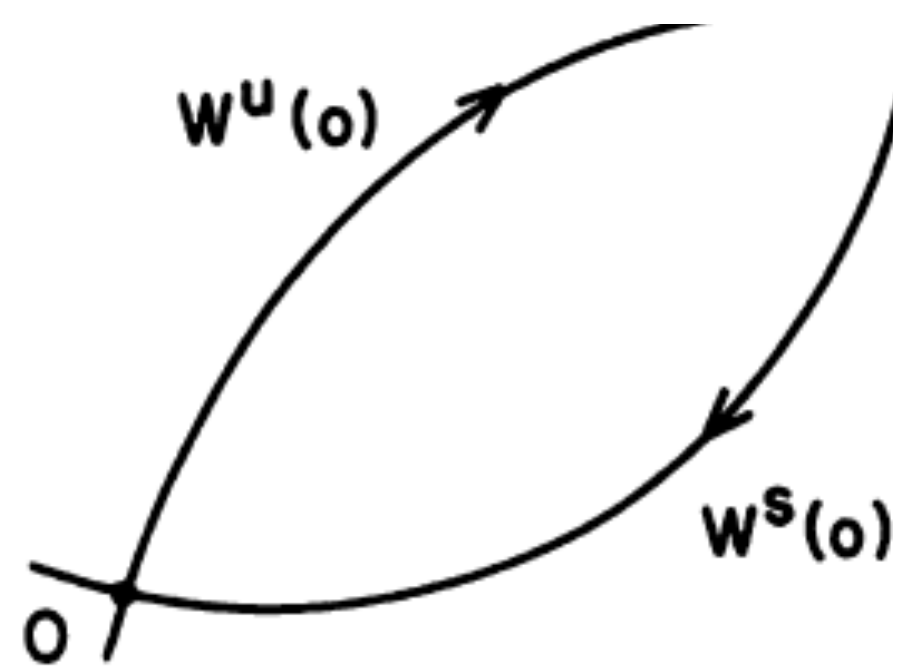
We conclude that $f$ on $\Lambda$ is chaotic.

# 3. TRANSVERSE HOMOCLINIC POINTS

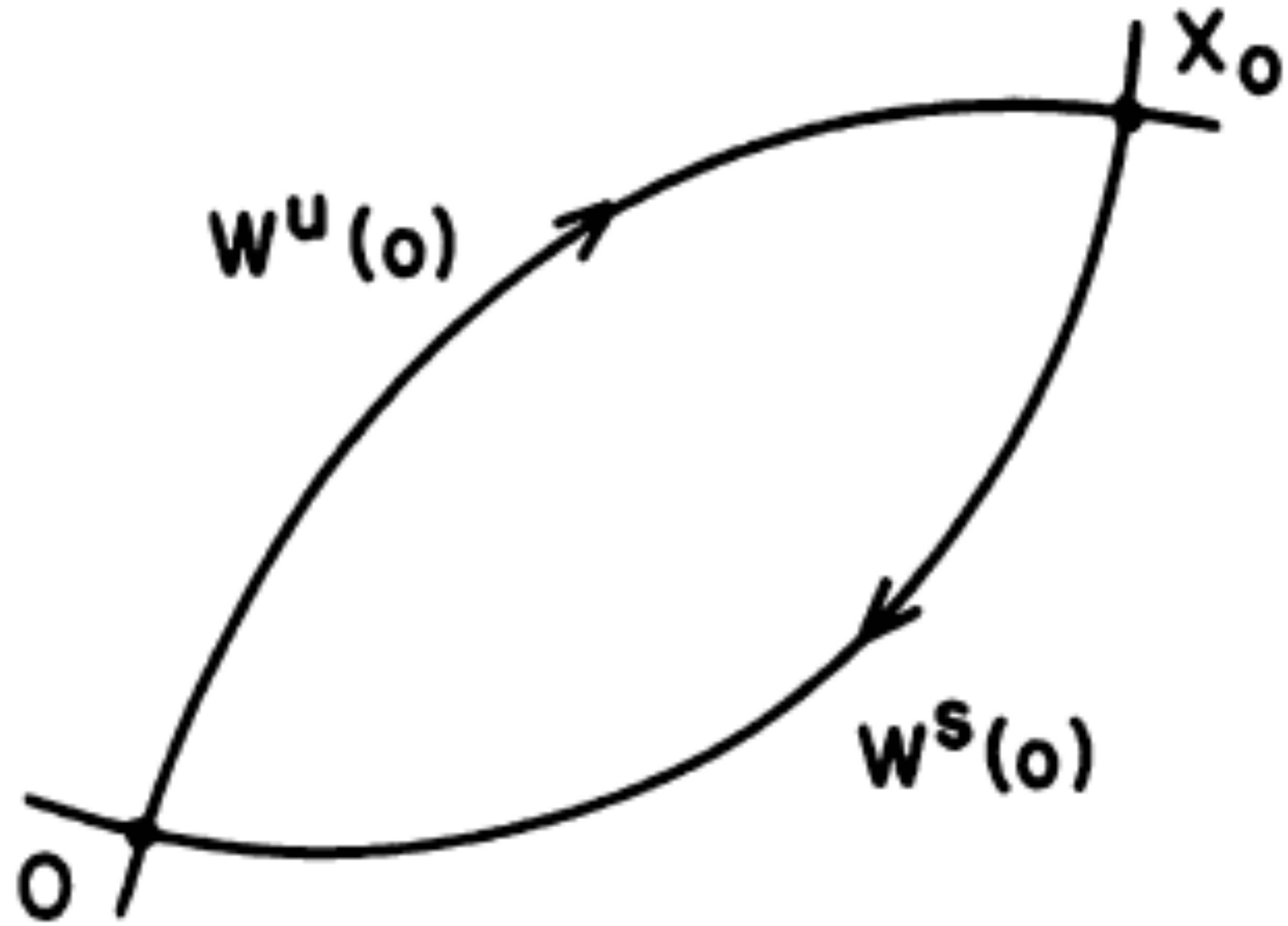
# HORSESHOE IN MAPS

Consider a diffeomorphism defining a map *P*
Assume it has a fixed point at 0

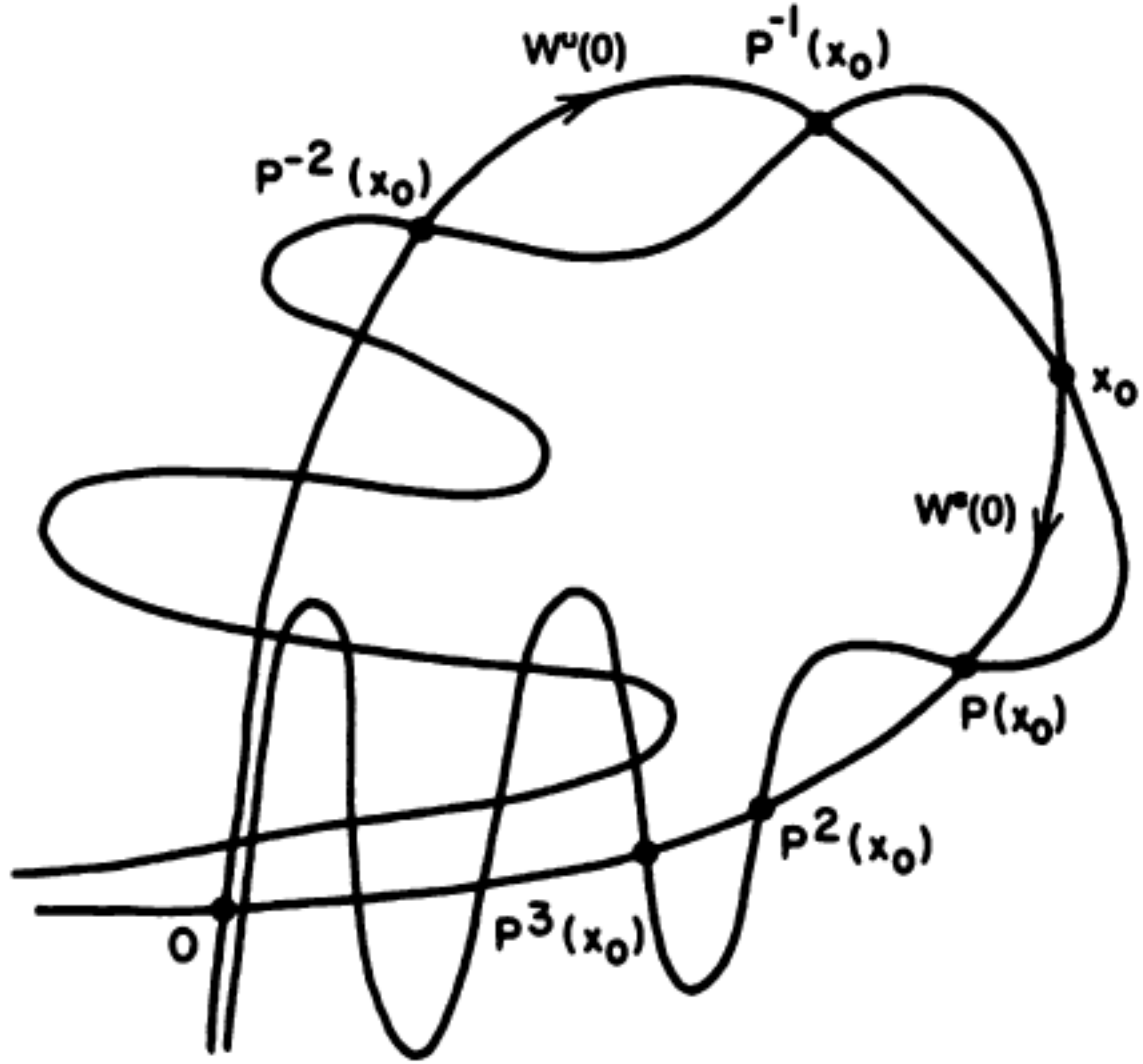
Build the stable and unstable manifolds to this point



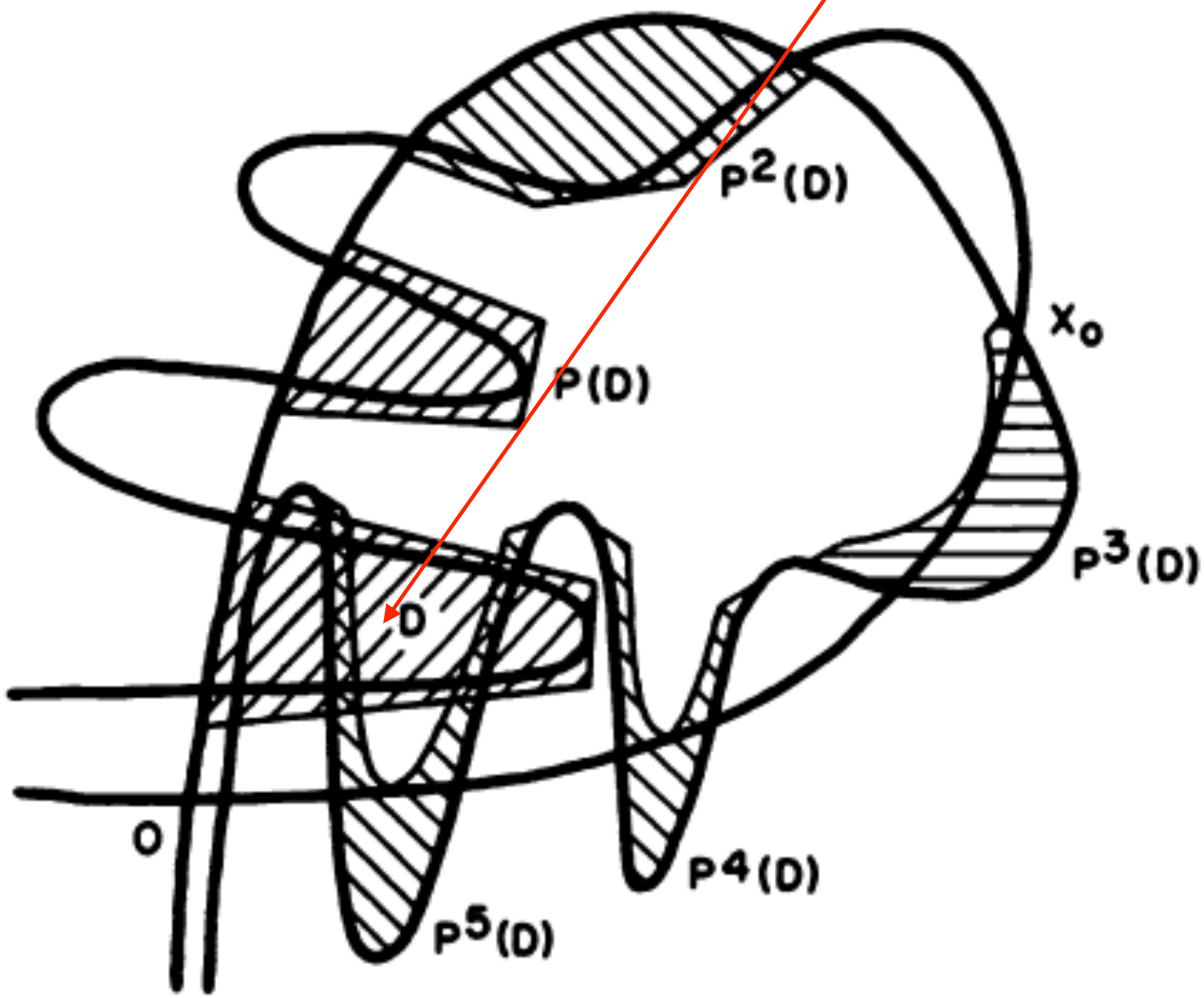
Assume that the stable and unstable manifold intersect transversally at a point $x_0$

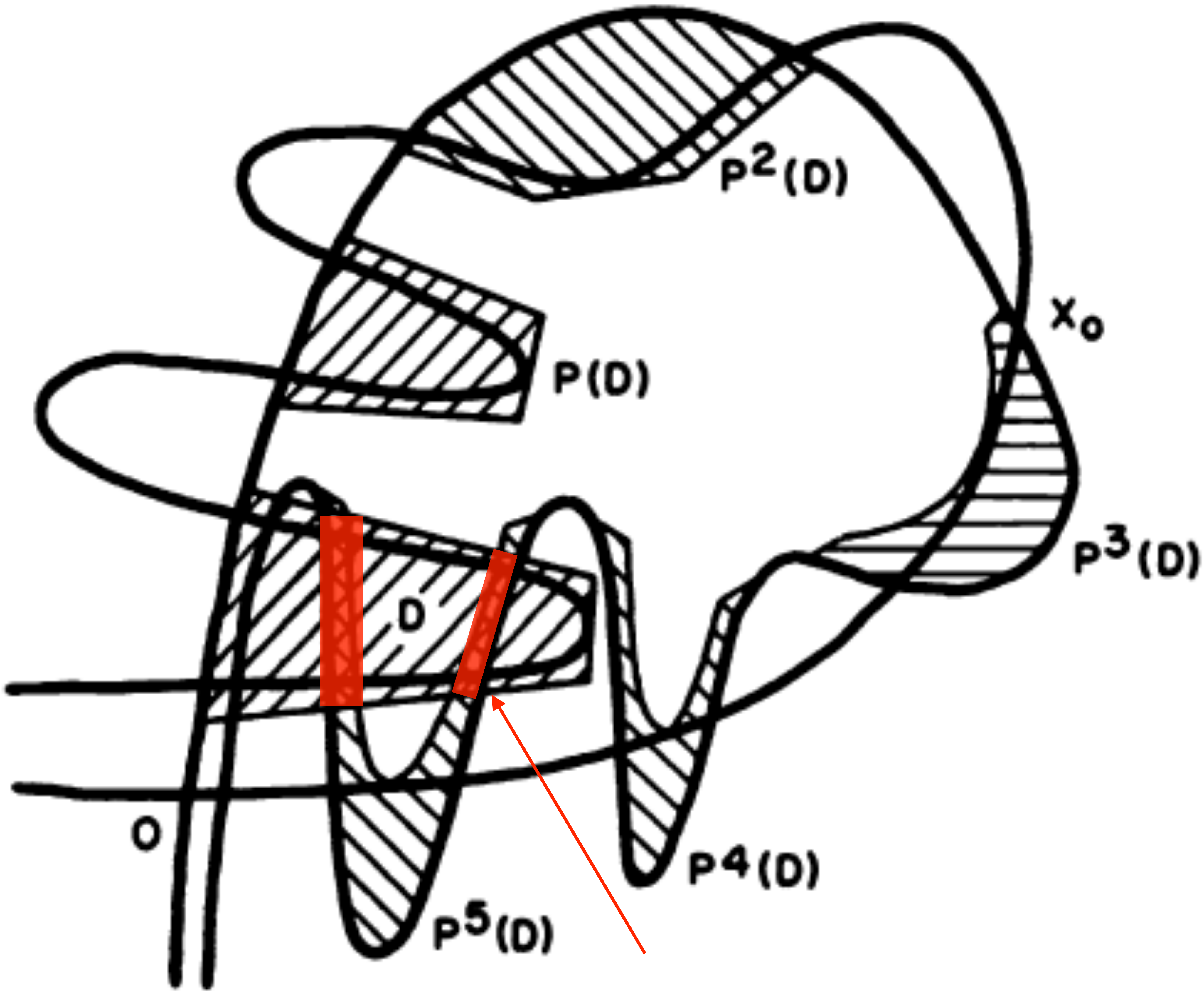


If these manifolds intersect once they MUST intersect infinitely many times



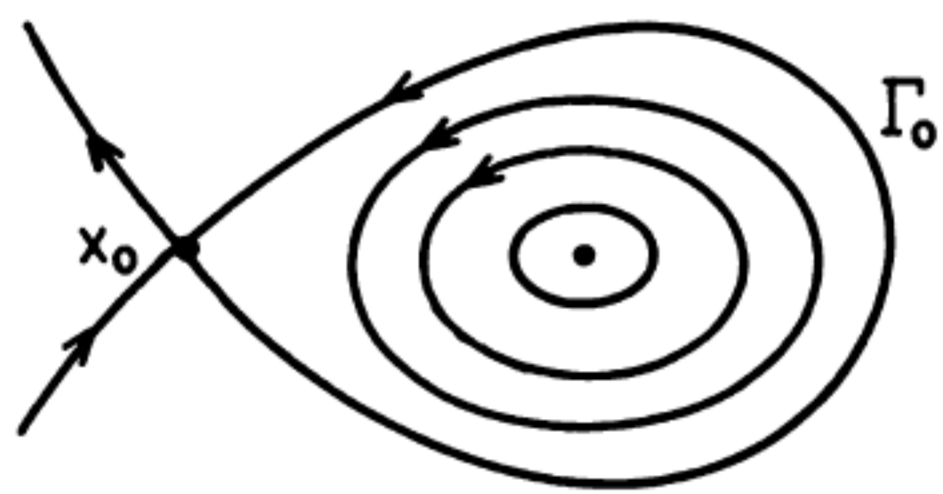
Now consider the evolution of a domain D defined by one of these homoclinic tangles



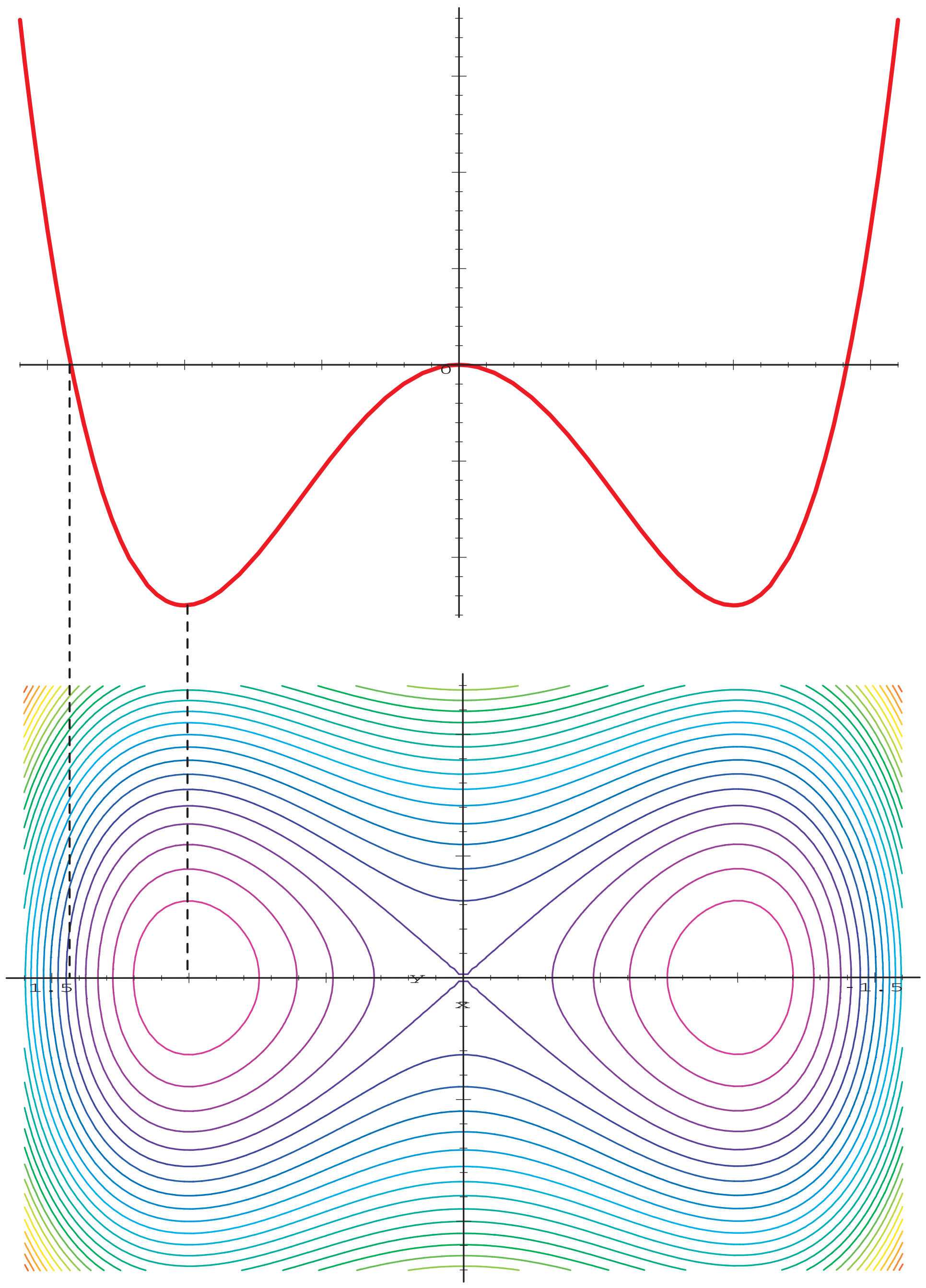



Intersection of D and $P^5$(D)

# 4. MELNIKOV's METHOD



# Duffing equation



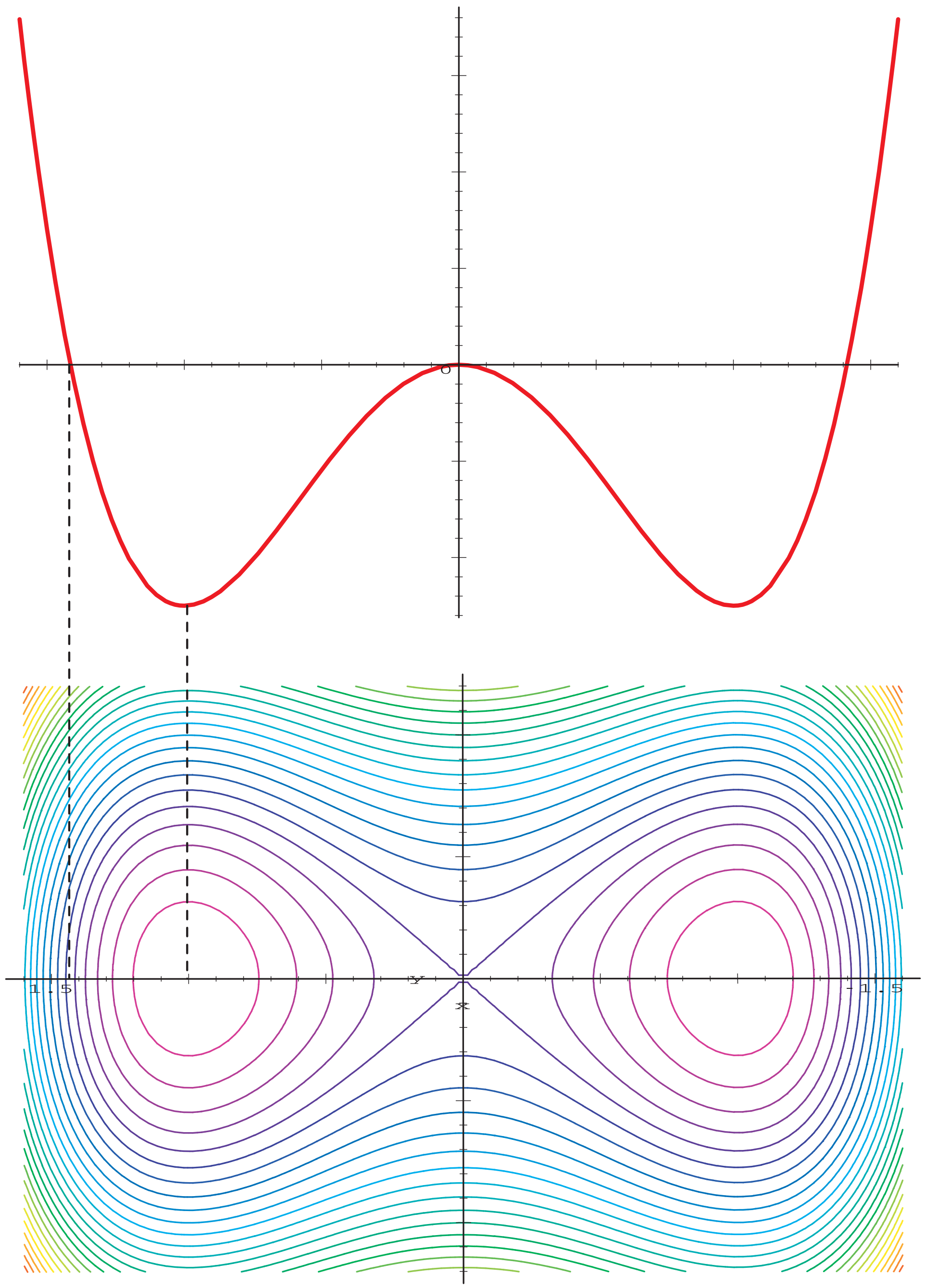
$$\ddot{x} = -\text{grad } V,$$

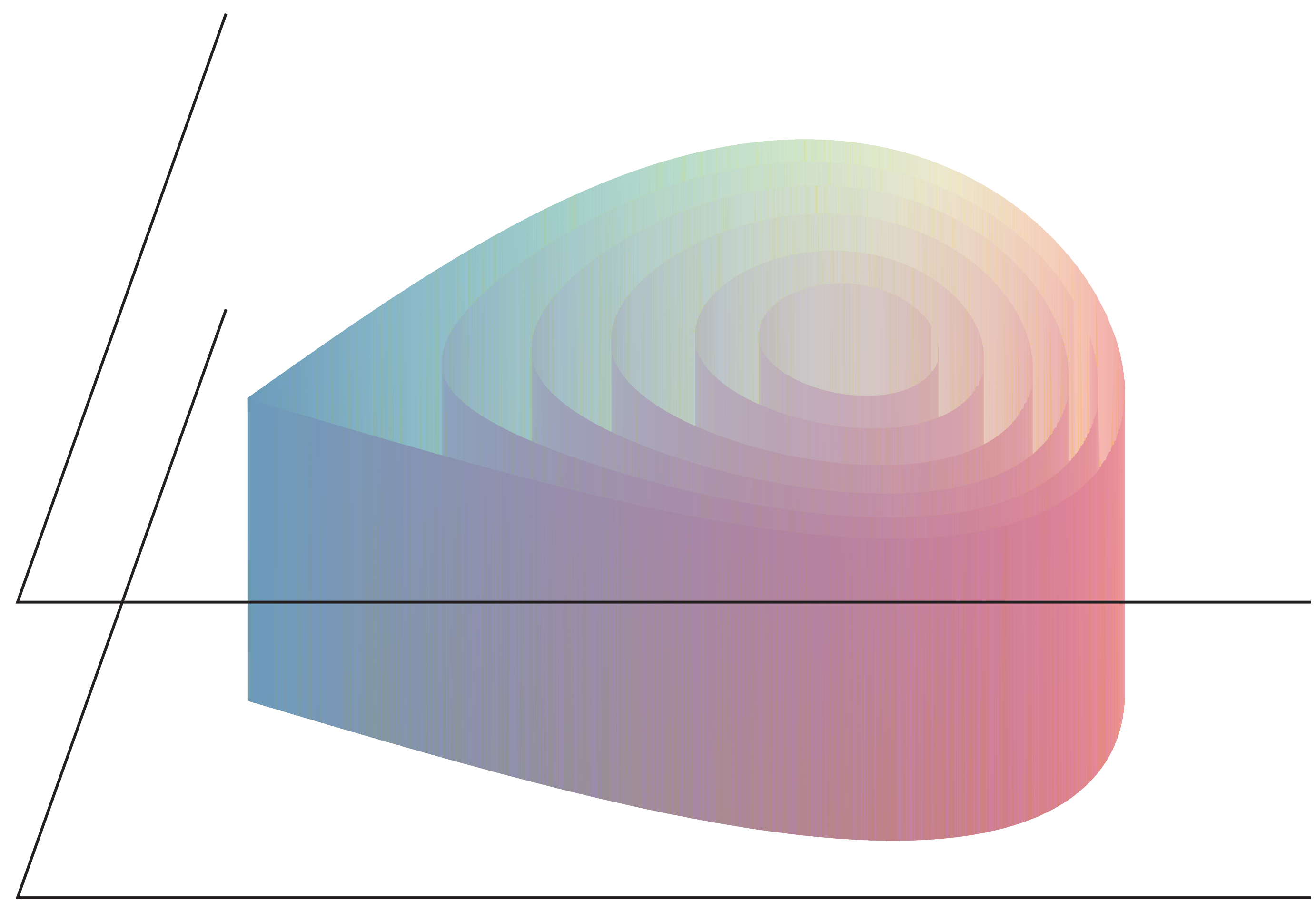
$$\ddot{x} - x + x^3 = 0.$$

# Duffing equation

$$\ddot{x} = -\text{grad } V,$$

$$\ddot{x} - x + x^3 = 0.$$

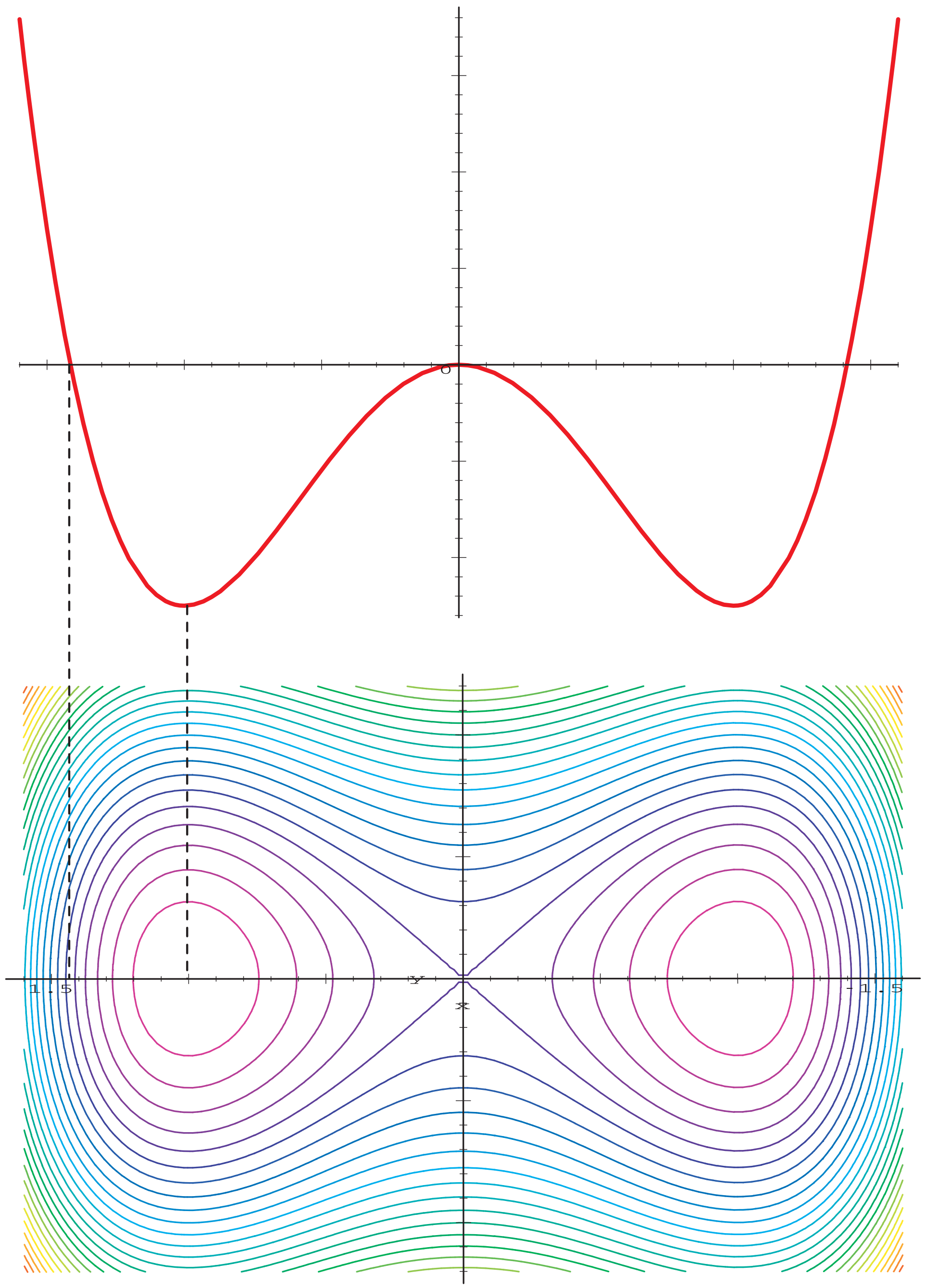


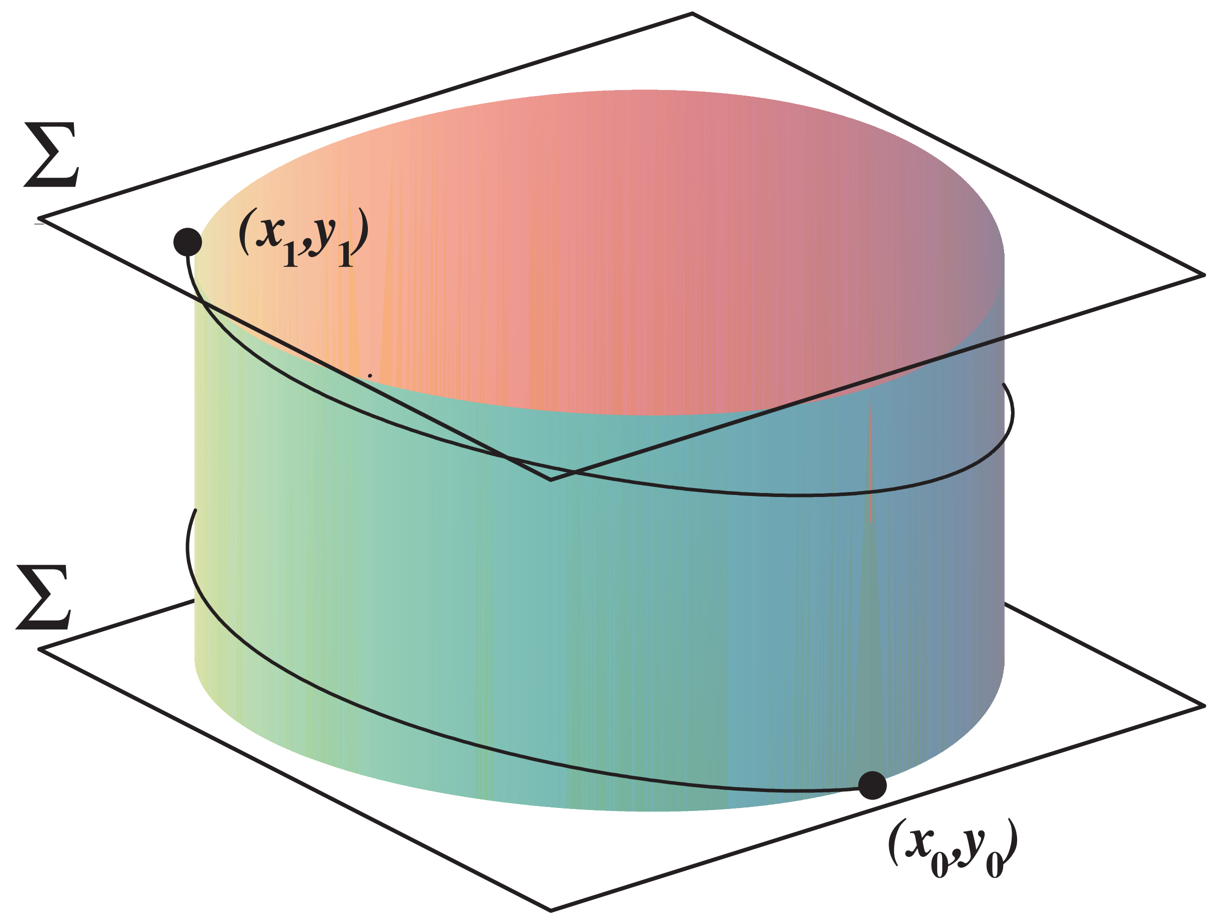


# Duffing equation

$$\ddot{x} = -\text{grad } V,$$

$$\ddot{x} - x + x^3 = 0.$$



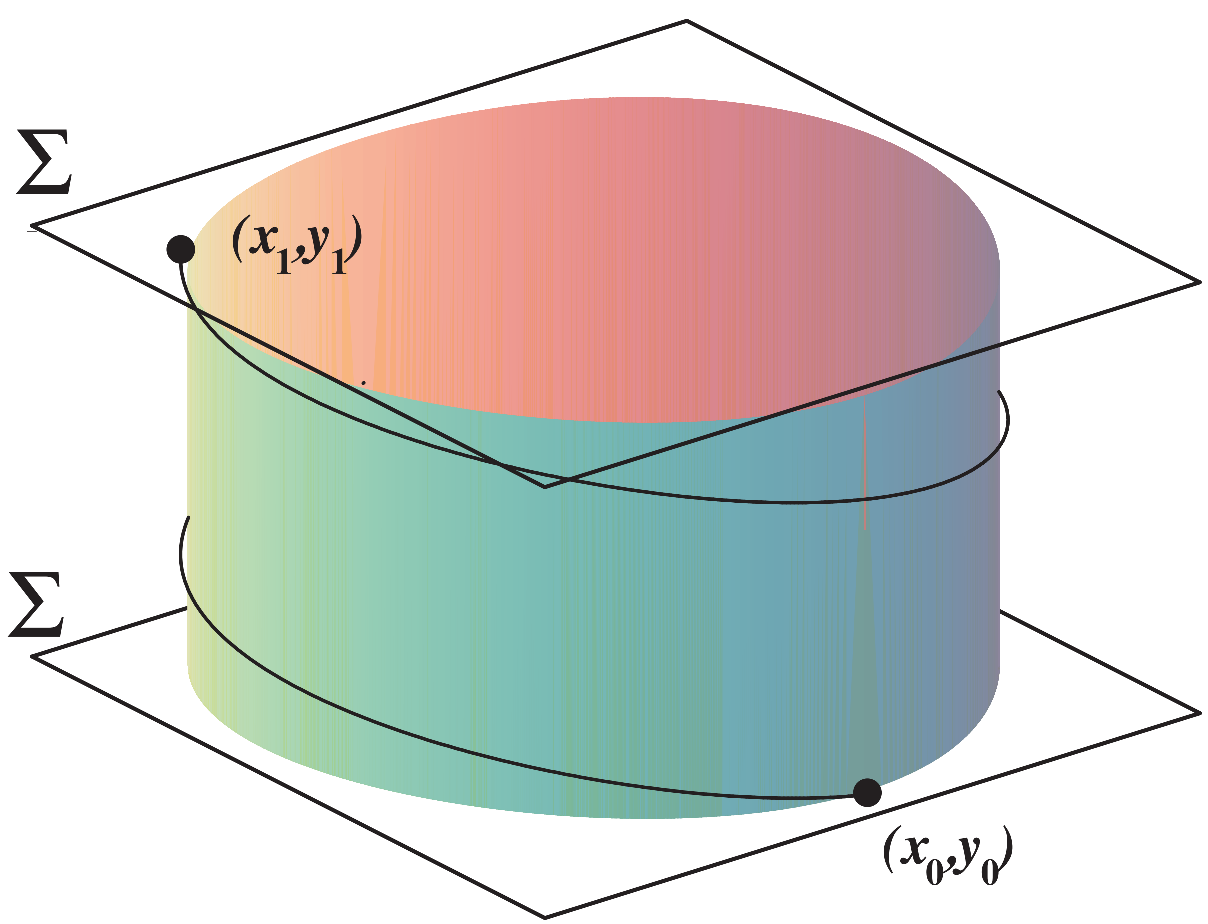


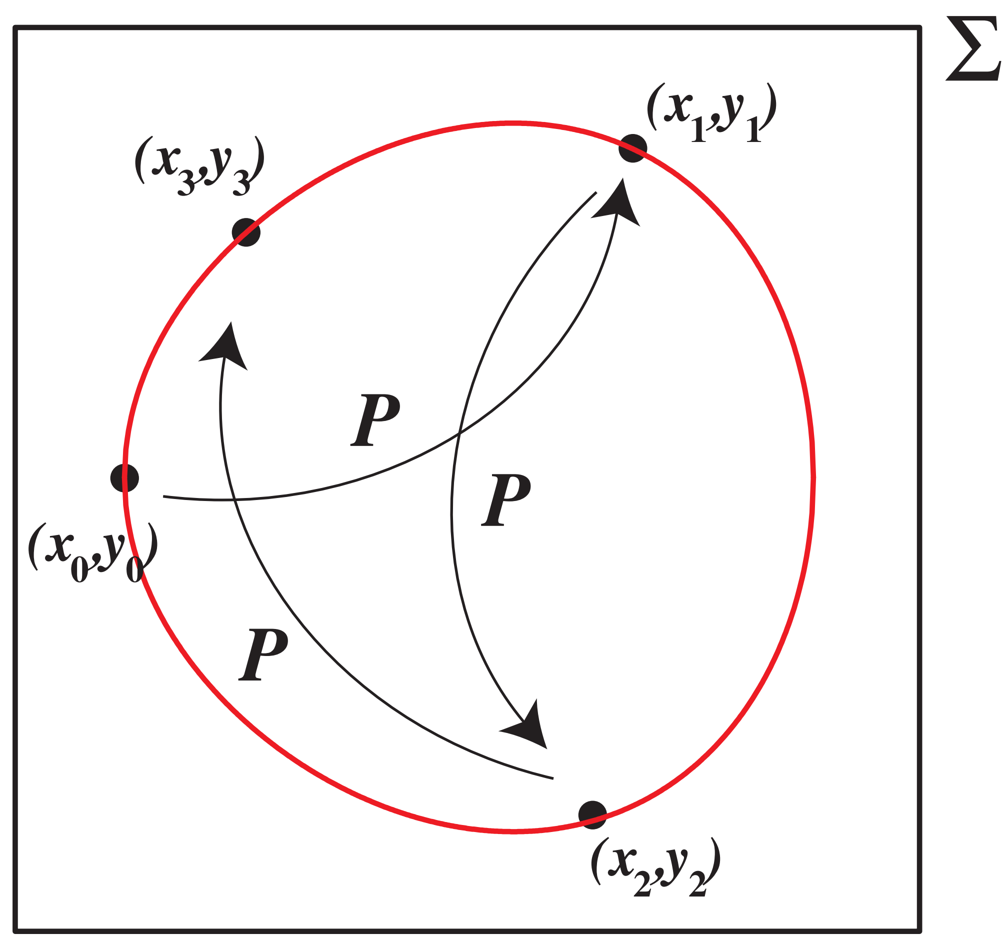
# Duffing equation

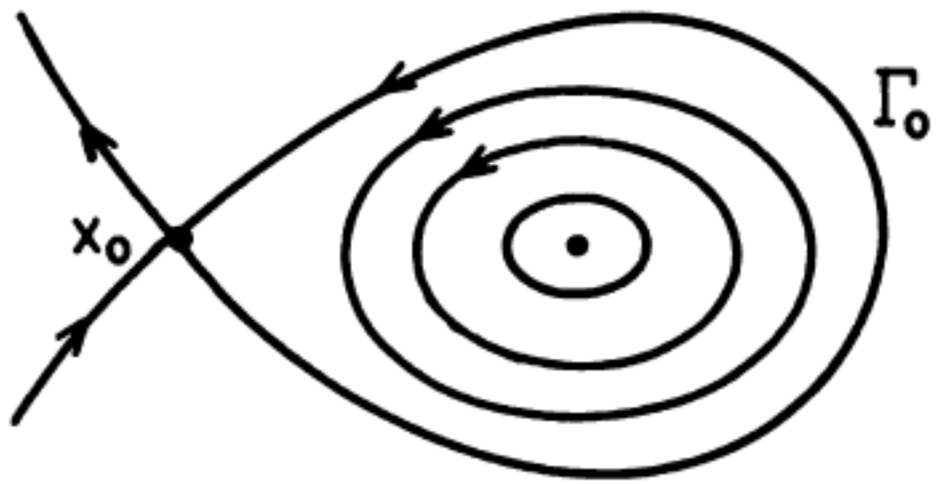
$$\ddot{x} = -\text{grad } V,$$

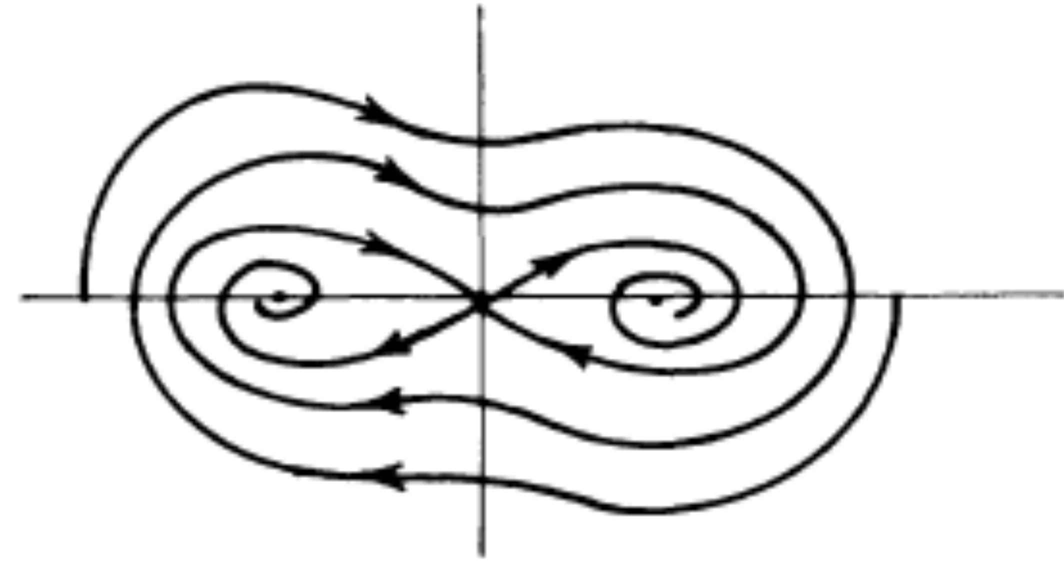
Poincaré map
for the unperturbed system

$$\ddot{x} - x + x^3 = 0.$$

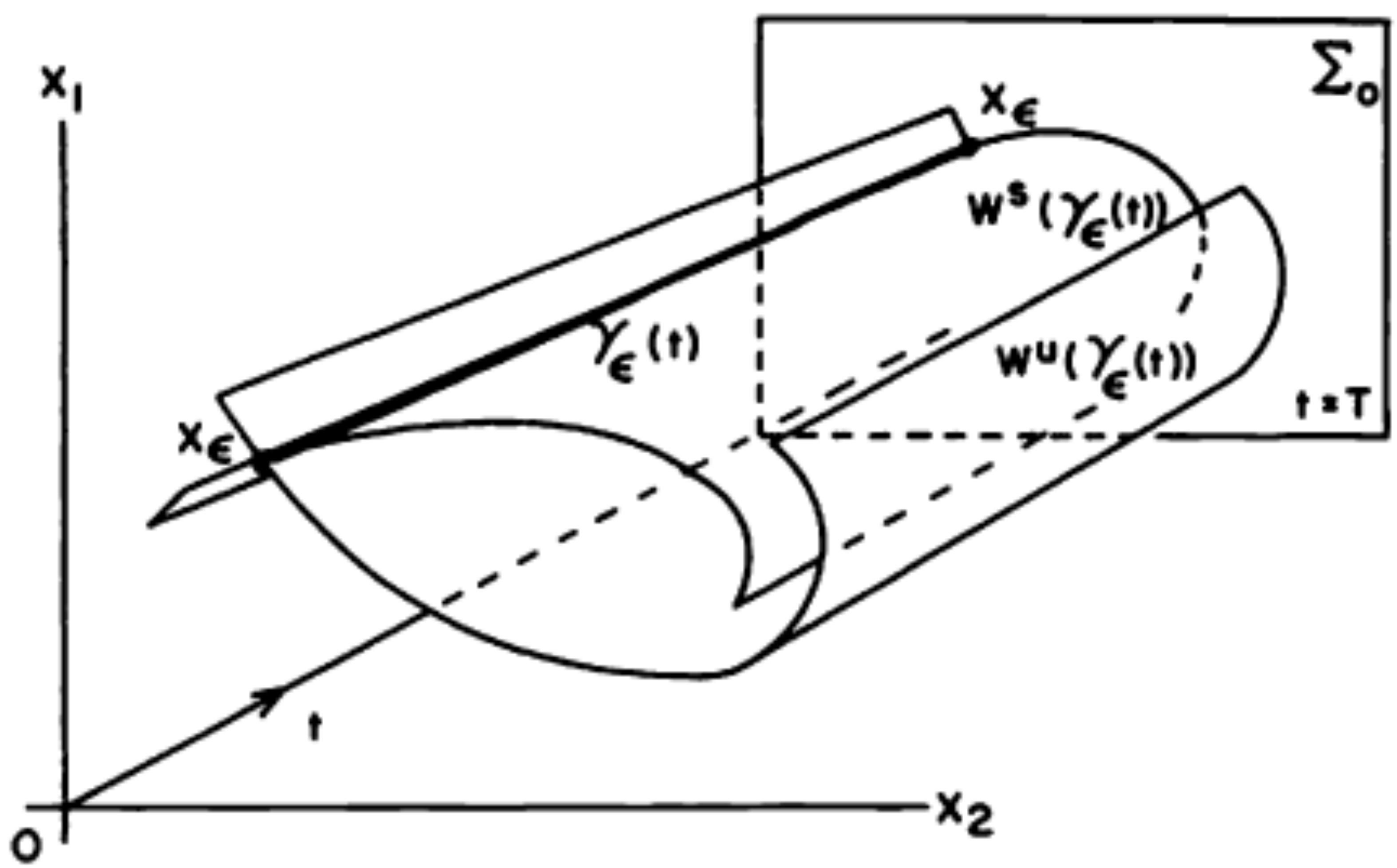


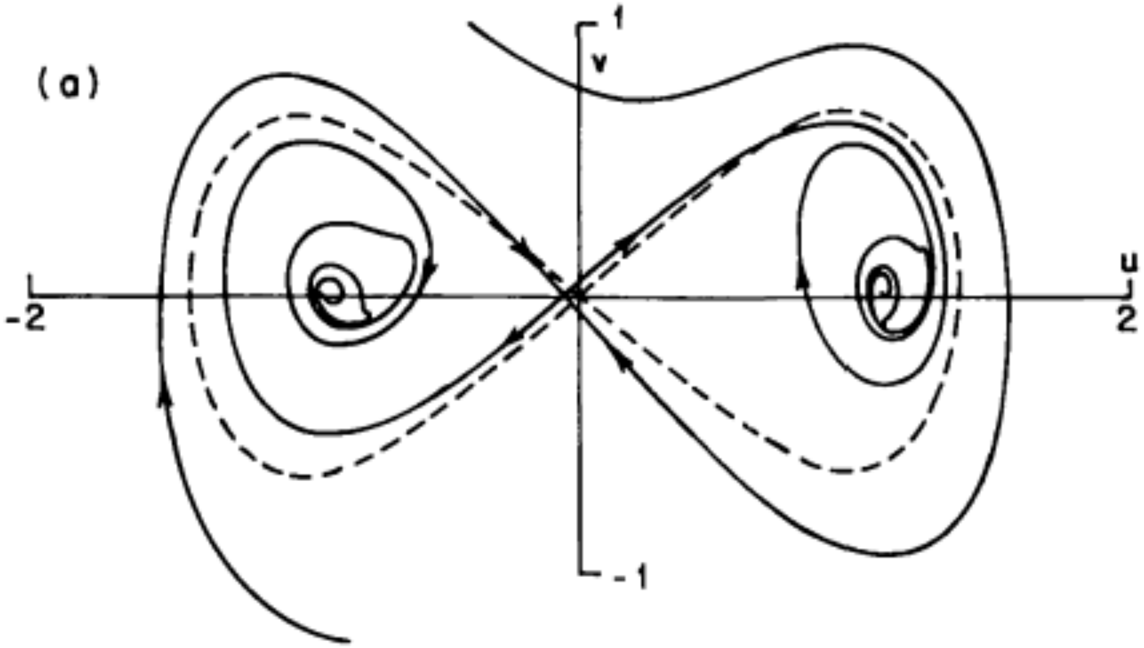


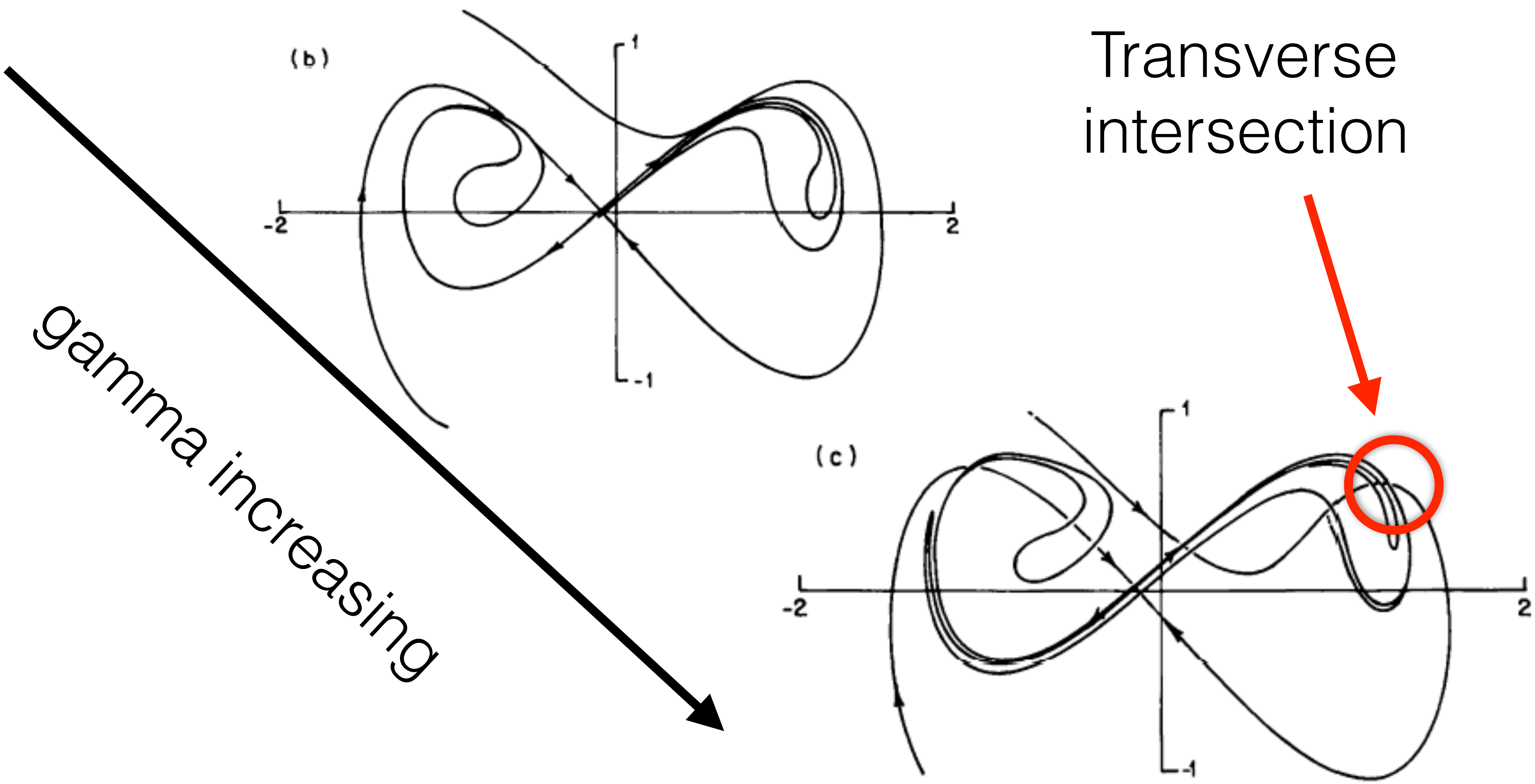



With perturbation

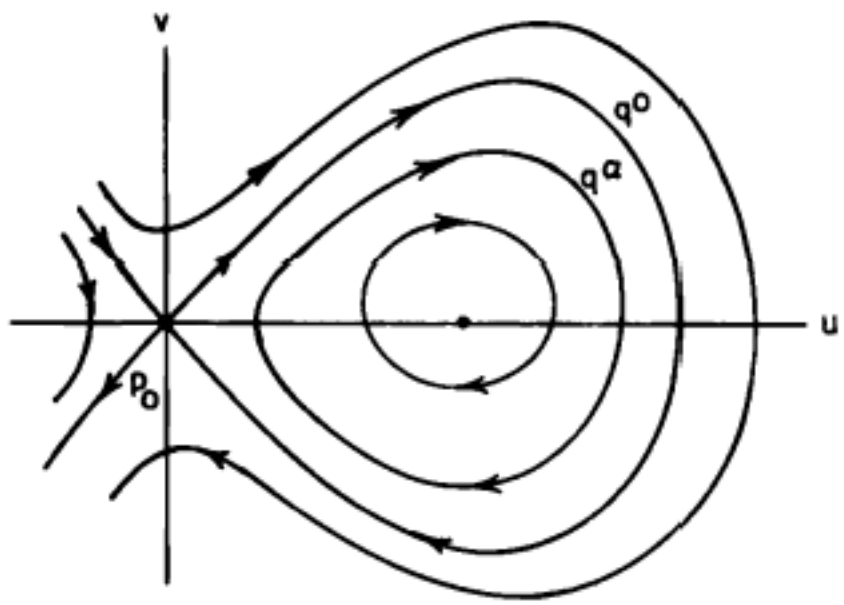


$$x''(t) + \delta\, x'(t) + x(t)^3 - x(t) = \gamma \cos(t)$$

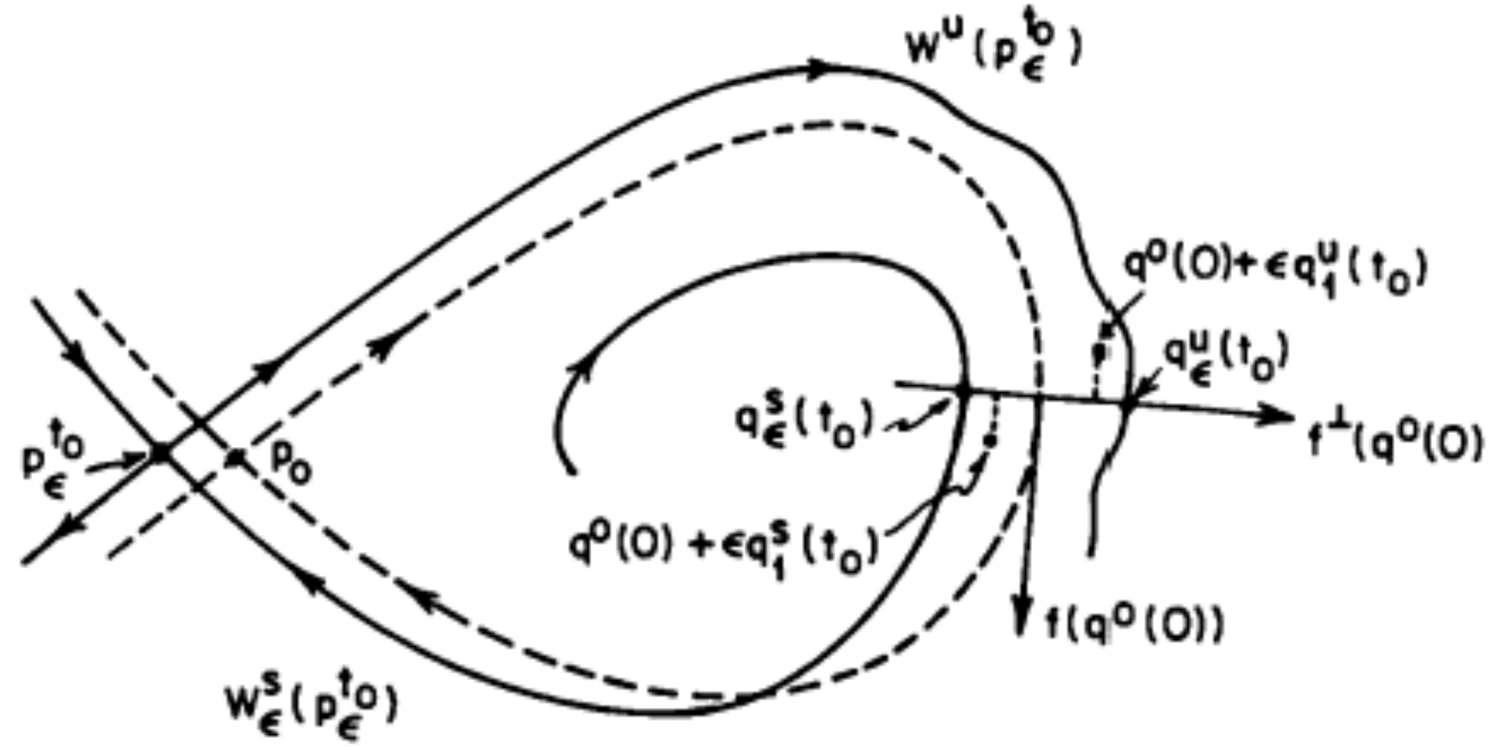


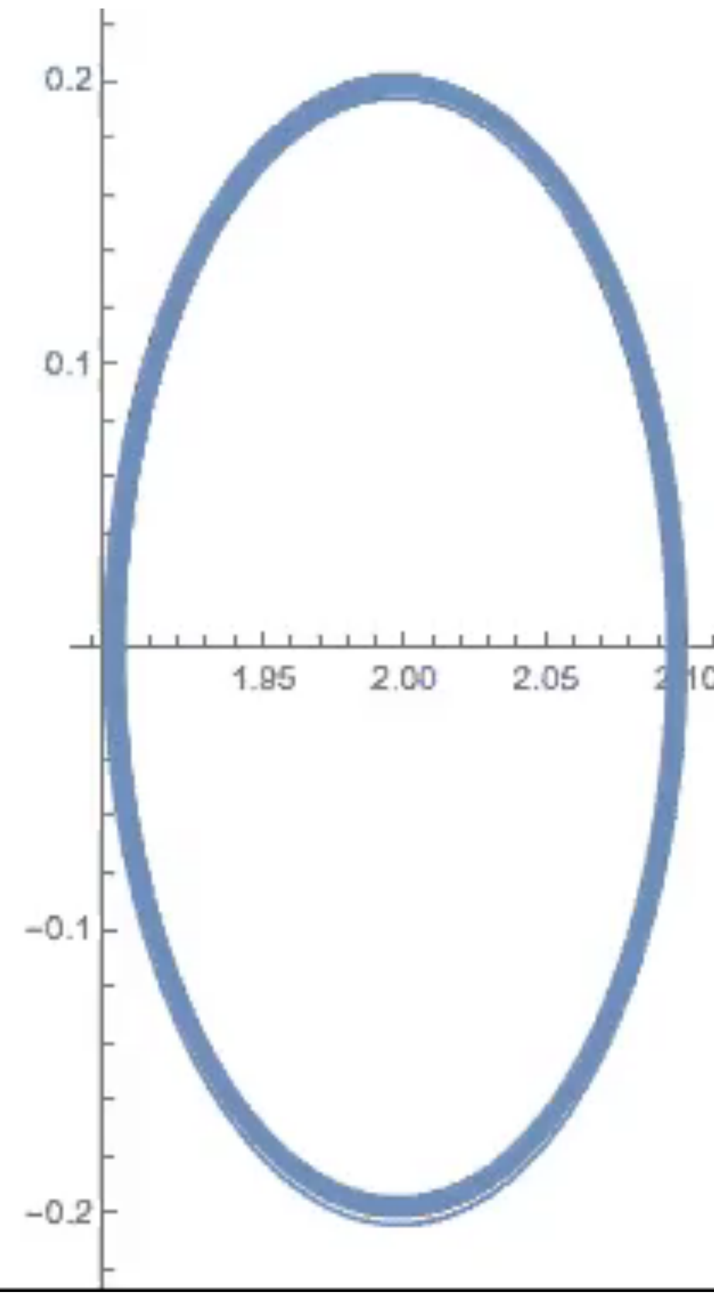
# Melnikov's method




Measure separation distance
between stable and unstable manifold

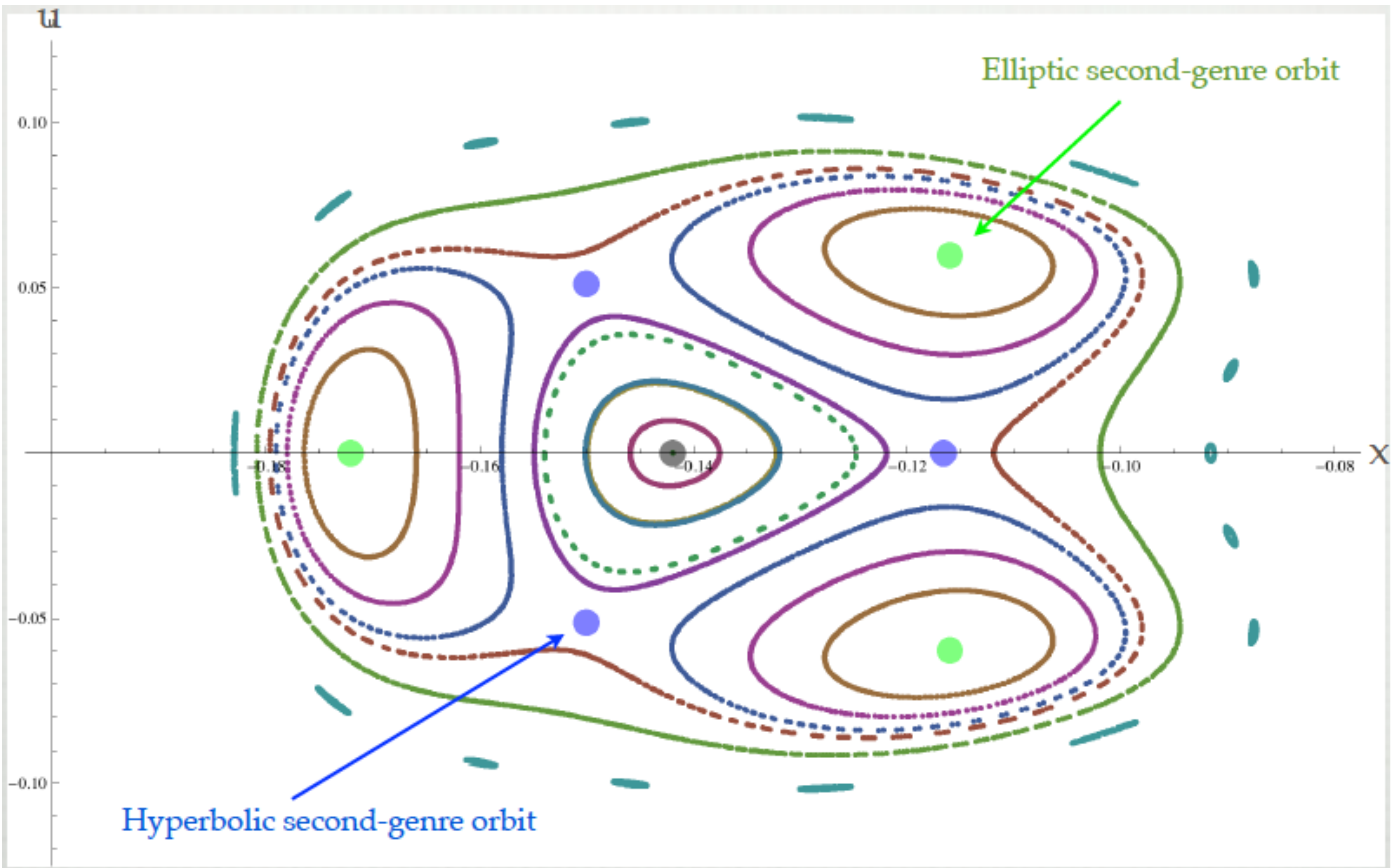


$$x''(t) + \delta\, x'(t) + x(t)^3 - x(t) = \gamma \cos(t)$$



# Transverse homoclinic in celestial mechanics

# Poincaré section



# Study guidelines

- Examinable material=synopsis (the one online)
- Methods:
  - Linear stability analysis (study of Jacobian matrix)
  - Centre manifold reduction (computing manifold and dynamics on manifold)
  - Lyapunov and first integrals (stability and level set)
  - One-degree-of-freedom system (drawing phase portrait from V)
  - bifurcation for flows and maps (identifying the bifurcation: saddle node, transcritical, pitchfork, hopf, period doubling)
  - analysis of one-dimensional maps (fixed point, periodic orbits)
  - Melnikov's method (computing transverse intersection)

# Study guidelines

- Examinable material=synopsis (the one online)
  Reproduced below:

**Course Synopsis:**

1. Geometry of linear systems
   Basic concepts of stability and linear manifold of solutions. Orbits in phase-space, linear flows, eigenvalues of fixed points.
2. Geometry on nonlinear systems
   Notion of flows, invariant sets, asymptotic sets, attractor. Conservative and Non-Conservative systems.
3. Local analysis
   Stable manifold theorem, notion of hyperbolicity, center manifold.
4. Bifurcation.
   Bifurcation theory: codimension one normal forms (saddle-node, pitchfork, trans-critical, Hopf).
5. Maps
   Poincaré sections and first-return maps. Stability and periodic orbits; bifurcations of one-dimensional maps, period-doubling.
6. *Chaos
   Maps: Logistic map, Bernoulli shift map, symbolic dynamics, Smale's Horseshoe Map. Melnikov's method. Differential equations: Lorenz equations, Rossler equations.

We have covered it all (except for the Rossler equations, which are not part or the examinable material )

# Study guidelines

- Examinable material=synopsis (the one online)
- Concepts and definition:
  - Linear spaces; stable, unstable, centre manifolds
  - Attractor and attracting sets, notions of stability
  - Linear stability theorem, hyperbolic fixed points
  - Notion of bifurcation
  - Poincaré maps, period doubling
  - Orbits: fixed point, periodic, limit cycle  non-periodic, homoclinic
  - Shift map, horseshoe, chaotic system