B5.6 Nonlinear Systems

3. Local analysis

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What we learned from Section 1&2.

- For a system of linear autonomous equations $\dot{\mathbf{x}} = A\mathbf{x}$, the solutions live on invariant space that can be classified according to the eigenvalues of A.
- The stable (resp. unstable, centre) linear subspace is the span of eigenvectors whose eigenvalues have a negative (resp. positive, null) real part.
- For nonlinear systems, we define the notion of asymptotic sets (α and ω limit set), the notion of attracting set, and basin of attraction.
- We define two important notions of stability for a fixed point: *(Lyapunov) stability* (i.e. "solutions remain close") and *exponential stability* i.e. ("fixed point is stable AND all nearby solutions converge to the fixed point asymptotically for long time").
- Lyapunov functions can be used to test stability. But, finding a Lyapunov function can be difficult.

3. Local analysis

3. Local analysis

3.1 The problem

The problem

Consider the nonlinear, autonomous, first-order system of differential equations with fixed point $\mathbf{x}_0$:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \qquad \text{and} \quad \mathbf{x}_0 \text{ such that } \mathbf{f}(\mathbf{x}_0) = \mathbf{0}. \tag{1}$$

with vector field $\mathbf{f} : E \subset \mathbb{R}^n \to \mathbb{R}^n$.

Problem:
What is the stability of this fixed point?
Can it be obtained algorithmically?

Main idea

The basic idea is to look at nearby solutions by expanding $\mathbf{x}$ close to $\mathbf{x}_0$

$$\mathbf{x} = \mathbf{x}_0 + \boldsymbol{\xi} \tag{2}$$

Inserting this in the equation and using the fact that $\mathbf{x}_0$ is a fixed point, we have

$$\begin{aligned} \dot{\boldsymbol{\xi}} &= \mathbf{f}(\mathbf{x}_0 + \boldsymbol{\xi}) && (3) \\ &= D\mathbf{f}(\mathbf{x}_0)\boldsymbol{\xi} + \mathcal{O}(|\boldsymbol{\xi}|^2) && (4) \end{aligned}$$

where $D\mathbf{f}(\mathbf{x}_0)$ is the *Jacobian matrix* associated with a vector field $\mathbf{f}$. It is the matrix of first derivatives evaluated at $\mathbf{x}_0$:

$$[D\mathbf{f}(\mathbf{x}_0)]_{ij} = \left[\frac{\partial f_i}{\partial x_j}\right]\Bigg|_{\mathbf{x}=\mathbf{x}_0} . \tag{5}$$

NB: Since $\mathbf{x}_0$ is constant $D\mathbf{f}(\mathbf{x}_0)$ is a matrix with constant coefficients.

Main idea

The local equations are

$$\dot{\boldsymbol{\xi}} = D\mathbf{f}(\mathbf{x}_0)\boldsymbol{\xi} + \mathcal{O}(|\boldsymbol{\xi}|^2) \tag{6}$$

The *variational equations* or *linearised equations* are given by the linear system obtained by dropping the nonlinear terms in the above equations:

$$\dot{\boldsymbol{\xi}} = D\mathbf{f}(\mathbf{x}_0)\boldsymbol{\xi} \tag{7}$$

This equation (7) is a linear equation with a constant matrix.
We know we can solve it and that the stability of $\boldsymbol{\xi} = \mathbf{0}$ is determined by $\mathrm{Spec}(D\mathbf{f}(\mathbf{x}_0))$.

The central problem of *local analysis* is to relate the stability of $\boldsymbol{\xi} = \mathbf{0}$ for (7) to the stability of $\boldsymbol{\xi} = \mathbf{0}$ for (6).

3. Local analysis

3.2 Stable manifold theorem

Basic idea in the plane

Consider the system

$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y) \end{cases} \tag{8}$$

and assume without loss of generality that $x = y = 0$ is a fixed point. The linearised system is

$$\begin{cases} \dot{\xi} = \partial_x f(0,0)\xi + \partial_y f(0,0)\eta \\ \dot{\eta} = \partial_x g(0,0)\xi + \partial_y g(0,0)\eta \end{cases} \tag{9}$$

We assume that the eigenvalues of the Jacobian matrix

$$D\mathbf{f}(0) = \begin{bmatrix} \partial_x f(0,0) & \partial_y f(0,0) \\ \partial_x g(0,0) & \partial_y g(0,0) \end{bmatrix} \tag{10}$$

are real (but non-vanishing) with opposite sign.

Stable manifold theorem

Locally, all the trajectories converging to the fixed point for large positive (resp. negative) time, define curves in the plane.

This curve is called the *local stable manifold* (resp. *local unstable manifold*).

These curves are locally tangent to the stable and unstable linear subspaces of the linear system. Explicitly, for this system, we define the stable and unstable manifolds as

$$W^s(0) = \{(x, y) \in \mathbb{R}^2 | \varphi_t(x, y) \xrightarrow[t\to\infty]{} 0\} \tag{11}$$

$$W^u(0) = \{(x, y) \in \mathbb{R}^2 | \varphi_t(x, y) \xrightarrow[t\to-\infty]{} 0\} \tag{12}$$

Stable manifold theorem

Theorem 3.1

Let $\varphi_t : E \subset \mathbb{R}^n \to E$ be the flow of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with fixed point $\mathbf{x}_0$. Suppose that the spectrum of $D\mathbf{f}(x_0)$ is composed of k eigenvalues with positive real parts and $(n-k)$ eigenvalues with negative real parts. Then,

- *there exists, in a neighbourhood of $\mathbf{x}_0$ a $(n-k)$-dimensional manifold $W^s_{loc}(x_0)$ tangent to E^s such that*

$$\forall\ \mathbf{x} \in W^s_{loc},\ t \geq 0,\ \varphi_t(\mathbf{x}) \xrightarrow[t\to\infty]{} \mathbf{x}_0. \tag{13}$$

- *there exists, in a neighbourhood of $\mathbf{x}_0$ a k-dimensional manifold $W^u_{loc}(x_0)$ tangent to E^u such that*

$$\forall\ \mathbf{x} \in W^u_{loc},\ t \leq 0,\ \varphi_t(\mathbf{x}) \xrightarrow[t\to-\infty]{} \mathbf{x}_0, \tag{14}$$

Moreover, W^s_{loc} and W^u_{loc} are as smooth as $\mathbf{f}$.

The stable and unstable manifolds

The existence of **local** stable and and unstable manifolds allows us to define **global** *stable and unstable manifolds* as follows:

$$W^s(\mathbf{x}_0) = \bigcup_{t\leq 0} \varphi_t\left(W^s_{\text{loc}}(\mathbf{x}_0)\right) \tag{15}$$

$$W^u(\mathbf{x}_0) = \bigcup_{t\geq 0} \varphi_t\left(W^u_{\text{loc}}(\mathbf{x}_0)\right) \tag{16}$$

The stable and unstable manifolds

Example:

$$\begin{cases} \dot{x} = -x - y^2 \\ \dot{y} = y + x^2 \end{cases} \tag{17}$$

A few important observations

NB1: W^s and W^u are not solution curves (they are unions of curves).

NB2: If $\mathbf{f}$ is analytic, it follows that W^s and W^u are also analytic.

NB3: However, if $\mathbf{f}$ is analytic, it does NOT follow that all solution curves are analytic.

NB4: If $W^s \cap W^u \neq \emptyset$, then $W^s \cap W^u$ is a *homoclinic manifold*.
The property of the homoclinic manifold is that any initial condition on the manifold ends up asymptotically for negative and positive time on the same fixed point.

Hyperbolicity and stability

Definition 3.2

If $\text{Re}(\lambda) \neq 0$ for all $\lambda \in \text{Spec}(D\mathbf{f}(\mathbf{x}_0))$, then $\mathbf{x}_0$ is an *hyperbolic fixed point.*

The stability of hyperbolic fixed points is fully determined by the linearisation of the vector field around the fixed point:

Theorem 3.3

If Re$(\lambda) < 0$ *for all* $\lambda \in$ *Spec*$(D\mathbf{f}(\mathbf{x}_0))$, *then* $\mathbf{x}_0$ *is asymptotically stable.*
If there exists $\lambda \in$ *Spec*$(D\mathbf{f}(\mathbf{x}_0))$ *s.t. Re*$(\lambda) > 0$, *then* $\mathbf{x}_0$ *is unstable.*

Necessity of hyperbolicity

Example: The nonlinearly damped harmonic oscillator

$$\ddot{x} + \epsilon x^2 \dot{x} + x = 0 \tag{18}$$

3. Local analysis

3.3 The centre manifold

Tangency to the linear subspace

Recall the construction of the stable and unstable manifolds:
They are defined locally as the unique manifolds tangent to the stable and unstable linear subspaces of the linearised equations.
Then the stable and unstable manifolds are defined as the evolution in (negative and positive respectively) time of these local manifolds.

What happens if one of the eigenvalues has zero real part?
In this case, the linearised equations have a non-empty centre subspace.

NAIVE IDEA: Define W^c_{loc} as the orbits tangent to E^c.
The problem with this idea is that this set is not unique.

Counter-example

(Counter)-example:

$$\begin{cases} \dot{x} = x^2 \\ \dot{y} = -y \end{cases} \tag{19}$$

Centre manifold theorem

The previous example shows that there is a unique curve with the same smoothness as the vector field. This is extended to the following

Theorem 3.4

Let $\varphi_t : E \subset \mathbb{R}^n \to E$ be the flow of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with fixed point $\mathbf{x}_0$ where $f \in C^r(E)$. Suppose that the spectrum of $D\mathbf{f}(\mathbf{x}_0)$ has k eigenvalues with zero real parts and $(n-k)$ eigenvalues with non-zero real parts.
Then, there exists, in a neighbourhood of $\mathbf{x}_0$ a unique k-dimensional manifold $W^c_{loc}(x_0)$ that is

- *tangent to E^c at x_0;*
- *of class C^r;*
- *invariant under the flow.*

Stable, unstable, and centre manifolds

We can combine the two manifold theorems.

Given a vector field $\mathbf{f} \in C^r(E)$, $E \subset \mathbb{R}^n$, $r \geq 1$.
A fixed point $\mathbf{x}_0$, and a set of eigenvalues $\Lambda = \mathrm{Spec}\,(D\mathbf{f}(\mathbf{x}_0))$.
We have

- k_s eigenvalues $\lambda \in \Lambda$ with $\mathrm{Re}(\lambda) < 0$, with linear subspace E^s,
- k_u eigenvalues $\lambda \in \Lambda$ with $\mathrm{Re}(\lambda) > 0$, with linear subspace E^u,
- k_c eigenvalues $\lambda \in \Lambda$ with $\mathrm{Re}(\lambda) = 0$, with linear subspace E^c,

with $k_s + k_u + k_c = n$. Then there exist

- a unique k_s-dimensional manifold W^s_{loc} tangent to E^s at $\mathbf{x}_0$,
- a unique k_u-dimensional manifold W^u_{loc} tangent to E^u at $\mathbf{x}_0$,
- a unique k_c-dimensional manifold W^c_{loc} **of class** C^r tangent to E^c at $\mathbf{x}_0$.

3. Local analysis

3.4 Reduction to the centre manifold

Reduction to the centre manifold

Consider again a fixed point $\mathbf{x}_0$. If the unstable manifold is non-empty, the fixed point is unstable.

The remaining case to study is when the unstable manifold is empty and the system has both a non-empty stable and centre manifold.

Question: What is the stability of a fixed point in the presence of a centre manifold.

Basic idea: The stability is governed by the dynamics on the centre manifold.

Reduction to the centre manifold

Without loss of generality we assume that the original system has been brought, by a linear transformation, to the canonical form:

$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} + \mathbf{f}(\mathbf{x}, \mathbf{y}), & \mathbf{x} \in \mathbb{R}^{\dim W^c} \\ \dot{\mathbf{y}} = B\mathbf{y} + \mathbf{g}(\mathbf{x}, \mathbf{y}) & \mathbf{y} \in \mathbb{R}^{\dim W^s} \end{cases} \tag{20}$$

where $(\mathbf{x}_0, \mathbf{y}_0) = (\mathbf{0}, \mathbf{0})$ is a fixed point (i.e. $\mathbf{f}(\mathbf{0}, \mathbf{0}) = \mathbf{0}$ and $\mathbf{g}(\mathbf{0}, \mathbf{0}) = \mathbf{0}$) and

$$\begin{cases} \text{Re}(\lambda) = 0 \ \ \forall \ \lambda \in \ \text{Spec}(A), \\ \text{Re}(\lambda) < 0 \ \ \forall \ \lambda \in \ \text{Spec}(B). \end{cases} \tag{21}$$

Note that we also assume that $\mathbf{f}$ and $\mathbf{g}$ are nonlinear at the origin (the Jacobian $\partial(\mathbf{f}, \mathbf{g})/\partial(\mathbf{x}, \mathbf{y})$ vanishes identically).

Reduction to the centre manifold

The main idea is to obtain a description of the centre manifold in terms of the variables $\mathbf{x}$. We posit:

$$\mathbf{y} = \mathbf{h}(\mathbf{x}), \tag{22}$$

and we look for a suitable function $\mathbf{h}(\mathbf{x})$.
It implies

$$\dot{\mathbf{y}} = D\mathbf{h}(\mathbf{x})\dot{\mathbf{x}}. \tag{23}$$

Hence:

$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} + \mathbf{f}(\mathbf{x}, \mathbf{h}(\mathbf{x})), \\ \dot{\mathbf{y}} = D\mathbf{h}(\mathbf{x})\dot{\mathbf{x}} = B\mathbf{h}(\mathbf{x}) + \mathbf{g}(\mathbf{x}, \mathbf{h}(\mathbf{x})). \end{cases} \tag{24}$$

The second equation is an equation for $\mathbf{h}(\mathbf{x})$:

$$D\mathbf{h}(\mathbf{x})\left(A\mathbf{x} + \mathbf{f}(\mathbf{x}, \mathbf{h}(\mathbf{x}))\right) = B\mathbf{h}(\mathbf{x}) + \mathbf{g}(\mathbf{x}, \mathbf{h}(\mathbf{x})). \tag{25}$$

Reduction to the centre manifold

$$D\mathbf{h}(\mathbf{x})\left(A\mathbf{x}+\mathbf{f}(\mathbf{x},\mathbf{h}(\mathbf{x}))\right)=B\mathbf{h}(\mathbf{x})+\mathbf{g}(\mathbf{x},\mathbf{h}(\mathbf{x})). \tag{26}$$

Close to the origin we can solve it by expanding **h** in Taylor series:

$$\mathbf{h}=\sum_{\mathbf{m},\ |\mathbf{m}|=2}^{|\mathbf{m}|=d}\mathbf{h}_{\mathbf{m}}\mathbf{x}^{\mathbf{m}}+\mathcal{O}(|\mathbf{x}|^{d+1}), \tag{27}$$

and solving for the coefficients $\mathbf{h}_{\mathbf{m}}$ (using the fact that two polynomials are equal for all values of $\mathbf{x}$ if their coefficients are the same).

Note: Here we use the multinomial formalism for a vector $\mathbf{x}=(x_1,\ldots,x_n)$ and postive integer vector $\mathbf{m}=(m_1,\ldots,m_n)$:

$$\mathbf{x}^{\mathbf{m}}=\prod_{i=1}^{n}x_i^{m_i} \tag{28}$$

Reduction to the centre manifold

Once **h** is known, it can be inserted into the first set of equations and we have

Theorem 3.5

The dynamics of (20) on its centre manifold W^c *at the origin is, for* $(\mathbf{x}, \mathbf{y})$ *close enough to the origin, given by the dynamics of*

$$\dot{\tilde{\mathbf{x}}} = A\tilde{\mathbf{x}} + \mathbf{f}(\tilde{\mathbf{x}}, \mathbf{h}(\tilde{\mathbf{x}})). \tag{29}$$

Reduction to the centre manifold

Example:

$$\begin{cases} \dot{x} = x^2 y - x^5 \\ \dot{y} = -y + x^2 \end{cases} \tag{30}$$

Close enough to the origin, the dynamics in the full space is well approximated by the dynamics on the centre manifold:

Theorem 3.6 (Shadowing)

Let $(\mathbf{x}_0, \mathbf{y}_0)$ *be close enough to the origin. Then for all* $(\mathbf{x}(t), \mathbf{y}(t))$ *based on* $(\mathbf{x}_0, \mathbf{y}_0)$*, there exists a solution* $\tilde{\mathbf{x}}(t)$ *such that*

$$\begin{cases} \mathbf{x}(t) = \tilde{\mathbf{x}}(t) + \mathcal{O}(e^{-\gamma t}), \\ \mathbf{y}(t) = \mathbf{h}(\tilde{\mathbf{x}}(t)) + \mathcal{O}(e^{-\gamma t}), \end{cases} \tag{31}$$

for some constant $\gamma > 0$.

The method

A step-by-step method: We start with a system

$$\dot{\mathbf{z}} = \mathbf{F}(\mathbf{z}), \qquad \mathbf{z} \in \mathbb{R}^p. \tag{32}$$

Assume that it has a fixed point at $\mathbf{z}_0$ (i.e. $\mathbf{F}(\mathbf{z}_0) = \mathbf{0}$).
Assume also that $M = D\mathbf{F}(\mathbf{z}_0)$ has $n > 1$ eigenvalues with zero real parts and $m > 1$ eigenvalues with negative real parts ($n + m = p$).

Note: We assume that there is no eigenvalue with positive real part (otherwise the fixed point is unstable).

The method

Step 1: Reduction to a canonical form: Introduce the new variables

$$\mathbf{z} = \mathbf{z}_0 + C\tilde{\mathbf{z}}, \tag{33}$$

where C is chosen such that

$$C^{-1}MC = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right]. \tag{34}$$

where the matrices A and B of respective dimension n and m are s.t.:

$$\begin{cases} \mathrm{Re}(\lambda) = 0 \ \ \forall\, \lambda \in \ \mathrm{Spec}(A), \\ \mathrm{Re}(\lambda) < 0 \ \ \forall\, \lambda \in \ \mathrm{Spec}(B). \end{cases} \tag{35}$$

After the change of variable, the new system in the variable $\tilde{\mathbf{z}} = (\mathbf{x}, \mathbf{y})$ is

$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} + \mathbf{f}(\mathbf{x}, \mathbf{y}), & \mathbf{x} \in \mathbb{R}^n \\ \dot{\mathbf{y}} = B\mathbf{y} + \mathbf{g}(\mathbf{x}, \mathbf{y}) & \mathbf{y} \in \mathbb{R}^m \end{cases} \tag{36}$$

and $(\mathbf{0}, \mathbf{0})$ is a fixed point.

The method

Step 2: Reduction to the centre manifold: We want to solve

$$D\mathbf{h}(\mathbf{x})\left(A\mathbf{x}+\mathbf{f}(\mathbf{x},\mathbf{h}(\mathbf{x}))\right)=B\mathbf{h}(\mathbf{x})+\mathbf{g}(\mathbf{x},\mathbf{h}(\mathbf{x})). \tag{37}$$

Close to the origin we expand $\mathbf{h}$ in Taylor series:

$$\mathbf{h}=\sum_{\mathbf{m},\ |\mathbf{m}|=2}^{|\mathbf{m}|=d}\mathbf{h}_{\mathbf{m}}\mathbf{x}^{\mathbf{m}}+\mathcal{O}(|\mathbf{x}|^{d+1}), \tag{38}$$

We first choose $d=2$.
Inserting this expansion into (37) and expanding $\mathbf{g}$ in power series, we obtain a linear set of equations for $\mathbf{h}_{\mathbf{m}}$.
If there is a non-trivial solution to this set of linear equations, we have the first nonlinear approximation of the centre-manifold.
Otherwise, we increase the value of d until we find a non-trivial solution.

The method

Step 3: Dynamics on the centre manifold: We insert the first non-zero approximation

$$\mathbf{h} = \sum_{\mathbf{m},\ |\mathbf{m}|=2}^{|\mathbf{m}|=d} \mathbf{h}_{\mathbf{m}} \mathbf{x}^{\mathbf{m}}, \tag{39}$$

into

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{f}(\mathbf{x}, \mathbf{h}(\mathbf{x})). \tag{40}$$

and, to order d, we obtain the polynomial system:

$$\dot{\mathbf{x}} = A\mathbf{x} + \sum_{\mathbf{m},\ |\mathbf{m}|=2}^{|\mathbf{m}|=d} \mathbf{f}_{\mathbf{m}} \mathbf{x}^{\mathbf{m}}. \tag{41}$$

This is still a nonlinear system but of reduced dimension $n < p$.
The hope is that it is sufficiently simple to be analysed by elementary means (direct integration, Lyapunov functions,...).

4. Mappings

Iterative maps

We are interested in *iterative maps*, characterised by discrete iterations of the form

$$\mathbf{x}_{n+1} = \mathbf{G}(\mathbf{x}_n), \tag{42}$$

where $\mathbf{x} \in E \subset \mathbb{R}^m$.
Equivalently, we write

$$\mathbf{x} \mapsto \mathbf{G}(\mathbf{x}), \tag{43}$$

We note that

$$\mathbf{x}_1 = \mathbf{G}(\mathbf{x}_0), \quad \mathbf{x}_2 = \mathbf{G}(\mathbf{x}_1) = \mathbf{G}^{(2)}(\mathbf{x}_0), \ \ldots, \ \mathbf{x}_n = \mathbf{G}^{(n)}(\mathbf{x}_0). \tag{44}$$

with $\mathbf{G}^{(n)}(\mathbf{x}_0) = \mathbf{G}(\mathbf{G}(\ldots\mathbf{G}(\mathbf{x}_0)))$.

Two cases: Either $\mathbf{G}^{-1}$ exists and is uniquely defined.
Or, more generally, we can look at system for which only forward dynamics is defined.

4. Mappings

4.1 Linear maps

Linear maps

We start with the linear case:

$$\mathbf{x}_{n+1} = B\mathbf{x}_n, \quad B \in \mathcal{M}_m(\mathbb{R}),\ n \in \mathbb{Z}^+, \mathbf{x}_0 \in \mathbb{R}^m. \tag{45}$$

If $0 \notin \text{Spec}(B)$, then B can be inverted and orbits are unique.

The point $\mathbf{x}_0$ is a fixed point for the system is a point that is mapped onto itself:

$$\mathbf{x}_0 = \mathbf{G}(\mathbf{x}_0), \tag{46}$$

so for the linear case, it is a solution of

$$\mathbf{x}_0 = B\mathbf{x}_0. \tag{47}$$

We see that $\mathbf{x}_0 = \mathbf{0}$ is always a fixed point and we are interested in its stability.

Linear maps

The stability of $\mathbf{x}_0 = \mathbf{0}$ is given by the spectral properties of B.
Let

$$\begin{cases} \lambda_j = a_j + ib_j,\ \lambda_j \in \mathrm{Spec}(B) \\ \mathbf{w}_j = \mathbf{u}_j + i\mathbf{v}_j,\ \mathbf{u}_j, \mathbf{v}_j \in \mathbb{R}^m \end{cases} \tag{48}$$

where $\mathbf{w}_j$ is a generalised eigenvector.

Then, we define, *the stable, unstable, centre linear subspaces* as

- $E^s = \mathrm{Span}(\mathbf{u}_j, \mathbf{v}_j |\ j \text{ s.t. } |\lambda_j| < 1)$ (stable linear subspace)
- $E^c = \mathrm{Span}(\mathbf{u}_j, \mathbf{v}_j |\ j \text{ s.t. } |\lambda_j| = 1)$ (centre linear subspace)
- $E^u = \mathrm{Span}(\mathbf{u}_j, \mathbf{v}_j |\ j \text{ s.t. } |\lambda_j| > 1)$ (unstable linear subspace)

Linear maps

The stable linear subspace defines contraction mappings:
Let $\mathbf{x}_0 \in E^s$ then $\exists \alpha < 1, c > 0$ such that

$$|\mathbf{x}_n| \leq c\alpha^n |\mathbf{x}_0| \tag{49}$$

NB: There is a natural correspondence between flows and maps.

Every linear flow defines a linear map.
Consider a linear flow with matrix A. Fix t and define $B = e^{tA}$, then

$$\varphi_t(x_n) : \mathbf{x}_n \to B\mathbf{x}_n. \tag{50}$$

However, the converse is not true (can you give a counter-example?).

4. Mappings

4.2 Stability of maps

Stability

A fixed point for a mapping is a point $\mathbf{x}_0 \in \mathbb{R}^m$, such that $\mathbf{G}(\mathbf{x}_0) = \mathbf{x}_0$.

Definition 4.1

A fixed point $\mathbf{x}_0 \in \mathbb{R}^n$ is *stable* if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\forall \mathbf{x} \in B_\delta(\mathbf{x}_0)$, $\mathbf{G}^{(n)}(\mathbf{x}) \in B_\epsilon(\mathbf{x}_0)$ for all $n \in \mathbb{Z}^+$.

Definition 4.2

A fixed point $\mathbf{x}_0 \in \mathbb{R}^m$ is *asymptotically stable* if it is stable and $\exists \delta > 0$ such that $\forall \mathbf{x} \in B_\delta(\mathbf{x}_0)$

$$\mathbf{G}^{(n)}(\mathbf{x}) \xrightarrow[n \to \infty]{} \mathbf{x}_0. \tag{51}$$

4. Mappings

4.3 Stable and unstable manifolds

Stable and unstable manifolds

Consider an iterative map $\mathbf{x}_{n+1} = \mathbf{G}(x_n)$, where $\mathbf{G} : E \subset \mathbb{R}^m \to E$ and such that $\mathbf{G}^{-1}$ exists on the same domain.

We define, in a neighbourhood U of a fixed point $\mathbf{x}_0$, the *local stable and unstable manifolds*.
If the linear stable manifold of the linearised system has dimension n_s.
Then, there exists a n_s-dimensional local stable manifold W^s_{loc}, tangent to stable linear subspace E^s such that

$$W^s_{\text{loc}} = \left\{ \mathbf{x} \in U | \mathbf{G}^{(n)}(\mathbf{x}) \to \mathbf{x}_0, n \to \infty; \mathbf{G}^{(n)}(\mathbf{x}) \in U \ \forall n > 0 \right\} \qquad (52)$$

Stable and unstable manifolds

Similarly If the linear unstable manifold of the linearised system has dimension n_u. Then there exists a n_u-dimensional local unstable manifold, tangent to unstable linear subspace such that

$$W^u_{\text{loc}} = \left\{ \mathbf{x} \in U | \mathbf{G}^{(n)}(\mathbf{x}) \to \mathbf{x}_0, n \to -\infty; \mathbf{G}^{(n)}(\mathbf{x}) \in U \ \forall n > 0 \right\} \tag{53}$$

Stable and unstable manifolds

By extension, we define the *stable and unstable manifold*:

$$W^s(\mathbf{x}_0) = \bigcup_{n \leq 0} \mathbf{G}^n \left(W^s_{\text{loc}}(\mathbf{x}_0) \right) \tag{54}$$

$$W^u(\mathbf{x}_0) = \bigcup_{n \geq 0} \mathbf{G}^n \left(W^u_{\text{loc}}(\mathbf{x}_0) \right) \tag{55}$$

Note: Stable and unstable manifolds are not trajectories but union of trajectories.

Stable and unstable manifolds

Example:

$$\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \tag{56}$$

where $(x, y) \in \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

5. Periodic orbits

Periodic orbits

For a continuous dynamical system $\mathbf{x} = \mathbf{f}(\mathbf{x})$, a periodic orbit Γ is a closed curve in phase space $E \subset \mathbb{R}^m$.
Let $\mathrm{d}(\mathbf{x}, \Gamma)$ be the distance between a point $\mathbf{x}$ and Γ. Given a closed curve we can define a *neighbourhood of size* δ as the set of points

$$U_\delta(\Gamma) = \{\mathbf{x} \in E | \mathrm{d}(\mathbf{x}, \Gamma) < \delta\} \tag{57}$$

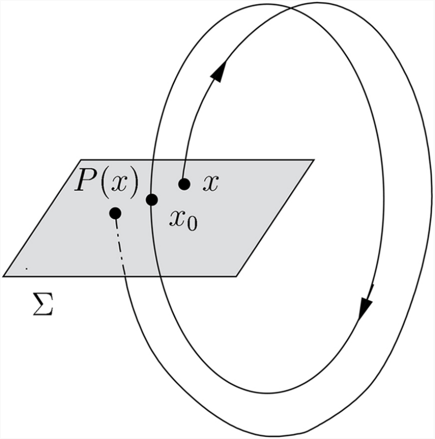
Definition 5.1

A periodic orbit Γ is *stable* if $\forall \epsilon > 0$, $\exists \delta > 0$ and a neighbourhood $U_\delta(\Gamma)$ such that $\forall \mathbf{x} \in U_\delta$, $\mathrm{d}(\varphi_t(\mathbf{x}), \Gamma) < \epsilon$.

From periodic orbits to maps

In a neighbourhood of a periodic orbit we can define the Poincaré map:

$$\mathbf{x} \mapsto \mathbf{P}(\mathbf{x}). \tag{58}$$



Stable and unstable manifolds

Example:

$$\begin{cases} \dot{r} = r(1 - r^2) \\ \dot{\theta} = 0 \end{cases} \tag{59}$$

Extra reading material

Extra material from the following books

[S] Strogatz, *Nonlinear Dynamics and Chaos with Applications to Physics, Biology, Chemistry and Engineering* (Westview Press, 2000).

[D] Drazin, *Nonlinear Systems* (Cambridge University Press, Cambridge, 1992).

[P] Perko, *Differential Equations and Dynamical Systems* (Second edition, Springer, 1996).

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