Nonlinear Systems

Problem Sheet 1

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■ **Note:** Unlike most courses, you are not required to do any problem (yes, you read this correctly!). You are now in charge of the way you approach your own learning. It is up to you to decide the problems you turn in. The following is a list of **suggested** problems for you to do to make sure you understand the concepts covered in class. If you think the list is too long, don't do them all. If you think it is too short, I am happy to provide more. Problems or sub-problems marked with a star* are meant to challenge you beyond the regular course. As with other problems, it is up to you to decide if you want to try them. I suggest that you give them a try and think about these problems as a way to gain a deeper understanding of the material.
In addition, you will find a companion file with detailed solutions. You are free to ignore these solutions, look at them, use them to check your own work, or to use them as hints if you are stuck on a problem. For your own learning, it may be a good idea to work out all the details for the solutions and write them as neatly as possible.

■ **Notations:** In all the problems, we will use the following (standard) notation: The dot over a symbol refers to the time derivative, so

$$\dot{x} \equiv \frac{\mathrm{d}x}{\mathrm{d}t}. \tag{1}$$

A vector in the n dimensional Euclidean space is represented in boldface $\mathbf{x} \equiv (x_1, x_2, \ldots, x_n)^\mathsf{T}$. Similarly, a *vector field* is $\mathbf{f}(\mathbf{x}) \equiv (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x}))^\mathsf{T}$. By default, a vector is a column vector, unless otherwise specified.

Matrices are indicated by uppercase. The *Jacobian matrix* $D\mathbf{f}$ associated with a vector field $\mathbf{f}$ is the matrix of first derivatives

$$[D\mathbf{f}]_{ij} = \left[\frac{\partial f_i}{\partial x_j}\right]. \tag{2}$$

Each component of the Jacobian matrix is a function of $\mathbf{x}$. Therefore, $D\mathbf{f}$ can be evaluated on a solution $\bar{\mathbf{x}}(t)$ which will be denoted

$$D\mathbf{f}(\bar{\mathbf{x}}(t)) = D\mathbf{f}|_{\mathbf{x}=\bar{\mathbf{x}}(t)}\,. \tag{3}$$

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1 Linear Systems

■ The goal of this section is to familiarise yourself with the basic notion of phase space, linear sub-spaces, solution of linear systems, and geometry.

1.1 Linear subspaces: Consider the system $\dot{\mathbf{x}} = A\mathbf{x}$ where $\mathbf{x} \in \mathbb{R}^3$ and

$$A = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 2 & 0 & 6 \end{bmatrix}. \tag{4}$$

Without solving the system, find the stable, unstable and center subspaces and sketch the phase portrait.

1.2 Linear subspaces: Let $A \in \mathcal{M}_n(\mathbb{R})$ be a semisimple matrix (i.e. a $n \times n$ matrix with real coefficients that can be diagonalised) and let $\mathbf{x} = \mathbf{x}(t)$ be a solution of

$$\dot{\mathbf{x}} = A\mathbf{x} \tag{5}$$
$$\mathbf{x}_0 = \mathbf{x}(t_0) \tag{6}$$

Show that:

(i) If $\mathbf{x}_0 \in E^s$, then $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$ and $\lim_{t\to-\infty} |\mathbf{x}(t)| = \infty$

(ii) If $\mathbf{x}_0 \in E^u$, then $\lim_{t\to\infty} |\mathbf{x}(t)| = \infty$ and $\lim_{t\to-\infty} \mathbf{x}(t) = \mathbf{0}$

(iii) If $\mathbf{x}_0 \in E^c$, then $\exists\ m, M \in \mathbb{R}$ such that $\forall\ t\ \in \mathbb{R}$:

$$m \leq |\mathbf{x}(t)| \leq M \tag{7}$$

(iv) * Which of these properties hold if A is not semisimple? (prove or give a counter-example)

1.3 *The variational equation and its adjoint: Consider the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{f}(\mathbf{x})$ is a C^1 vector field. Let $\bar{\mathbf{x}} = \bar{\mathbf{x}}(t)$ be a particular solution and consider the linear system obtained by looking at a nearby solution $\mathbf{x} = \bar{\mathbf{x}}(t) + \epsilon\mathbf{u}$:

$$\dot{\mathbf{u}} = D\mathbf{f}(\bar{\mathbf{x}})\mathbf{u} \tag{8}$$

(i) Show that $\dot{\bar{\mathbf{x}}}(t)$ is a solution of the linear system, and that it represents the tangent vector along the orbit.
(ii) For planar flows ($n = 2$), use this property to give a complete solution of this system of linear equations.

2 Nonlinear systems

■ Here, we study simple nonlinear features of nonlinear systems, such as invariant sets, homoclinic and heteroclinic orbits.

2.1 A heteroclinic orbit. A *heteroclinic orbit* is an orbit that connects two fixed points. Find the value of α such that the system

$$\dot{x} = x - y, \tag{9}$$
$$\dot{y} = -\alpha x + \alpha xy, \tag{10}$$

admits the first integral $I = (y - 2x + x^2)e^{-2t}$. (A scalar function $I = I(\mathbf{x}, t)$ is a *first integral* if $\dot{I} = 0$ on all trajectories.) Compute the fixed points and show that a branch of the level set of this first integral is a heteroclinic orbit. Can you find a closed form solution of this orbit?

2.2 Invariant set. The system

$$\dot{x} = -x \tag{11}$$
$$\dot{y} = -y + x^2 \tag{12}$$
$$\dot{z} = z + x^2 \tag{13}$$

defines a flow $\varphi_t : \mathbb{R}^3 \to \mathbb{R}^3$. Show that the set $S = \left\{(x, y, z) \in \mathbb{R}^3 | z = -x^2/3\right\}$ is an invariant set of this flow. Sketch this set in phase space and identify other interesting orbits (such as fixed points).

2.3 Attracting set. The system

$$\dot{x} = -y + x(1 - z^2 - x^2 - y^2) \tag{14}$$
$$\dot{y} = x + y(1 - z^2 - x^2 - y^2) \tag{15}$$
$$\dot{z} = 0 \tag{16}$$

defines a flow $\varphi_t : \mathbb{R}^3 \to \mathbb{R}^3$. Show that the union of the unit sphere with the portions of the z-axis outside the sphere is an attracting set for this flow. Find its domain of attraction. (Hint: rewrite the system in cylindrical coordinates).

2.4 Attracting set. Consider the system

$$\dot{r} = r(1 - r) \tag{17}$$
$$\dot{\theta} = \sin^2 \frac{\theta}{2}, \tag{18}$$

where (r, θ) are the usual polar coordinates of a point in the plane. Show that this system has two fixed points. Show that the fixed point $(x = 1, y = 0)$ is the ω-limit set of almost all initial conditions. That is $\varphi_t(\mathbf{x}_0) \to (1, 0))$ for all initial conditions $\mathbf{x}_0 \neq (0, 0)$. Despite that, show that $(1, 0)$ is not stable. Is it an attracting set? Is the unit circle an attracting set? Find the domain of attraction (if any).

2.5 Simple pendulum. The equation for the simple pendulum is

$$\ddot{x} + \sin x = 0, \quad x \in \mathbb{R}$$

Find the potential for this system and use it to identify important orbits. In particular identify the fixed points and show that there exist heteroclinic orbits for this system. Sketch the phase portrait. Show that the orbits contained within a symmetric pair of heteroclinic orbits (called a *heteroclinic cycle*) form an invariant set. Is this an attracting set? (This equation can be solved in terms of Elliptic integrals of the first kind, but in this case you are asked to answer all these questions without solving the equation explicitly).

3 Linearisation

■ One of the main tools of dynamical system is the linearisation of a nonlinear system close to its fixed points. From Part A, you should be familiar with the basic ideas. These exercises are meant as a refresher from last year.

3.1 A few warm-up problems. Consider the systems below. Find the fixed points and determine their stability through linearisation whenever possible. For systems with parameters, discuss stability with respect to the parameters

$$\dot{x} = 2x - 2xy \tag{19}$$
$$\dot{y} = 2y - x^2 + y^2 \tag{20}$$

$$\dot{x} = -4y + 2xy - 8 \tag{21}$$
$$\dot{y} = -x^2 + 4y^2 \tag{22}$$

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0 \tag{23}$$

3.2 Kolmogorov systems. Consider the two systems

$$\dot{x} = f(x, y) \tag{24}$$
$$\dot{y} = g(x, y) \tag{25}$$

and

$$\dot{x} = xf(x, y) \tag{26}$$
$$\dot{y} = yg(x, y), \tag{27}$$

where f and g are both C^1 functions.
(i) Show that for the second system, the first positive quadrant (defined as the set $S = \{(x, y) \in \mathbb{R}^2 | x > 0, y > 0\}$) is an invariant set.
(ii) Show that if the first system has an exponentially stable or unstable fixed point in S of the form (x_0, y_0) with $x_0 = y_0$, then the second one admits the same fixed point with the same stability properties (a fixed point is *exponentially stable* (resp. unstable) if the real parts of all its eigenvalues are strictly negative (resp. positive)). Is this property true for fixed points in the other quadrants (assuming $|x_0| = |y_0|$)? (Prove or give a counter-example)
*(iii)** Are these properties true for the similar problem in n dimensions. Formulate the corresponding results.