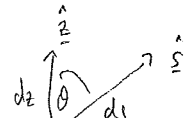
Sheet 1

1. Two stream approximation.



$dz = \cos\theta\, ds$

Radiative transfer equation $\boxed{\frac{\partial I}{\partial s} = -\kappa\rho\,(I - B)}$ for intensity of radiation travelling in direction $\hat{s}$

B = average intensity (local radiative equilibrium) $= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} I \sin\theta \, d\theta\, d\phi = \frac{1}{2}(I_+ + I_-)$ [using definition below].

$= \frac{\sigma T^4}{\pi}$ [from integrating Planck's function].

Writing in terms of z, $\frac{\partial}{\partial s} = \cos\theta \frac{\partial}{\partial z} = \mu \frac{\partial}{\partial z}$, so $\boxed{\mu \frac{\partial I}{\partial z} = -\kappa\rho\,(I - B) \qquad \mu = \cos\theta}$

For the two stream approximation, define $I_+ = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\frac{\pi}{2}} I \sin\theta\, d\theta\, d\phi = \int_0^1 I\, d\mu$.

$I_- = \frac{1}{2\pi}\int_0^{2\pi}\int_{\frac{\pi}{2}}^{\pi} I \sin\theta\, d\theta\, d\phi = \int_{-1}^{0} I\, d\mu$.

then approximate $\int_0^1 \mu \frac{\partial I}{\partial z}\, d\mu \approx \frac{1}{2}\frac{dI_+}{dz}$ and $\int_{-1}^{0} \mu \frac{\partial I}{\partial z}\, d\mu \approx -\frac{1}{2}\frac{dI_-}{dz}$, and $F_+ = 2\pi\int_0^1 \mu I_+\, d\mu \approx \pi I_+$

$F_- = 2\pi\int_{-1}^{0} \mu I_-\, d\mu \approx \pi I_-$

so the radiative transfer equation is integrated to give.

$$\frac{1}{2}\frac{dI_+}{dz} \approx -\kappa\rho\,(I_+ - B) = -\frac{1}{2}\kappa\rho\,(I_+ - I_-)$$
$$-\frac{1}{2}\frac{dI_-}{dz} \approx -\kappa\rho\,(I_- - B) = -\frac{1}{2}\kappa\rho\,(I_- - I_+)$$

Subtracting one equation from the other, we see $\frac{d}{dz}(I_+ - I_-) = 0$ so $I_+ - I_- = \frac{F}{\pi}$ is constant.

Hence $\frac{dI_+}{dz} = -\kappa\rho \frac{F}{\pi}$ and $\frac{dI_-}{dz} = -\kappa\rho\frac{F}{\pi}$.

with boundary conditions $\boxed{\begin{array}{l} I_- = 0 \text{ at } z = \infty \\ I_+ = \frac{\sigma T_s^4}{\pi} \text{ at } z = 0 \end{array}}$ (top of atmosphere) (Stefan-Boltzmann law for $F_+ = \pi I_+$).

Integrating $\Rightarrow$ $I_- = \frac{F}{\pi}\int_z^{\infty} \kappa\rho\, dz = \frac{F\tau}{\pi}$ and hence $I_+ = \frac{F}{\pi}(1+\tau)$

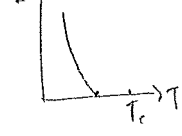
surface boundary condition $\Rightarrow$ $\sigma T_s^4 = F(1 + \tau_s)$ $\Rightarrow$ $F = \frac{\sigma T_s^4}{1 + \tau_s}$.

Since $F = \sigma T_e^4$ this means $\boxed{T_e^4 = \frac{T_s^4}{1 + \tau_s}}$ so T_s is larger than T_e, and increases with increasing τ_s.

The air temperature is given by $B = \frac{1}{2}(I_+ + I_-) = \frac{F}{\pi}\left(\frac{1}{2} + \tau\right) = \frac{\sigma T_e^4}{\pi}\left(\frac{1}{2} + \tau\right)$ $\Rightarrow$ $\boxed{T = T_e\left(\frac{1}{2} + \tau\right)^{1/4}}$

"$\frac{\sigma T^4}{\pi}$"

so the ground air temperature $T|_{z=0} = T_e\left(\frac{1}{2} + \tau_s\right)^{1/4} < T_e(1 + \tau_s)^{1/4} = T_s$.



2. Runaway greenhouse effect.

Clausius Clapeyron equation $\quad \dfrac{\partial p_{sv}}{\partial T} = \dfrac{\rho_v L}{T} \qquad p_{sv} = p_{sv0}$ at $T = T_0$.

$$= \frac{p_{sv}}{T^2}\frac{M_v L}{R} \qquad \left[\text{using } p_{sv} = \frac{\rho_v R T}{M_v}\right]$$

[integrate] $\Rightarrow \quad \dfrac{dp_{sv}}{p_{sv}} = \dfrac{M_v L}{R}\dfrac{dT}{T^2}$

$$\ln \frac{p_{sv}}{p_{sv0}} = \frac{M_v L}{R}\left\{\frac{1}{T_0} - \frac{1}{T}\right\}.$$

$$\Rightarrow \boxed{p_{sv} = p_{sv0}\, e^{a\left(1 - \frac{T_0}{T}\right)}} \qquad \boxed{a = \frac{M_v L}{R T_0}}$$

If $T - T_0 \ll T_0$ write $1 - \dfrac{T_0}{T} = 1 - \dfrac{1}{1 + \frac{T - T_0}{T_0}} = 1 - \left(1 - \dfrac{T - T_0}{T_0} + \dots\right) \doteq \dfrac{T - T_0}{T_0} + O\left(\dfrac{T - T_0}{T_0}\right)^2$

Energy balance $\quad \frac{1}{4} Q = \sigma \gamma T^4 \quad \Rightarrow \quad \boxed{T = \left(\dfrac{Q}{4\sigma\gamma}\right)^{1/4} = \left(\dfrac{Q}{4\sigma}\right)^{1/4}\left(1 + b\left(\dfrac{p_v}{p_{sv0}}\right)^c\right)}$

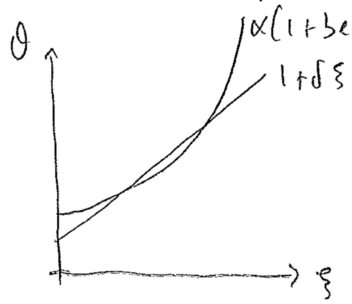
Write $\boxed{\theta = \dfrac{T}{T_0}}$, then energy balance gives $\quad \theta = \underbrace{\left(\dfrac{Q}{4\sigma T_0^4}\right)^{1/4}}_{\alpha}\left(1 + b\left(\dfrac{p_v}{p_{sv0}}\right)^c\right).$

$$\Rightarrow \boxed{\theta = \alpha\left(1 + b e^{\xi_v}\right)} \quad \text{defining} \quad \boxed{\xi_v = c \ln \frac{p_v}{p_{sv0}}}$$

With the same notation, the saturation curve becomes. $\quad \dfrac{\xi_{sv}}{c} = a(\theta - 1)$

(but $\xi_{sv} = c \ln \dfrac{p_{sv}}{p_{sv0}}$) $\quad \Rightarrow \boxed{\theta = 1 + \delta \xi_{sv}} \qquad \delta = \dfrac{1}{ac}.$

The runaway greenhouse effect occurs if the temperature calculated from energy balance remains above the saturation curve for all p_v (ie ξ_v). So its occurrence depends on the non-intersection of $\theta = \alpha(1 + b e^{\xi})$ with $\theta = 1 + \delta \xi$.



Clearly from the graph, non intersection will happen if α is large enough.
The critical α is found from when the curves meet tangentially, ie.
$$\boxed{\begin{aligned} \alpha(1 + b e^{\xi}) &= 1 + \delta \xi \\ \& \quad \alpha b e^{\xi} &= \delta. \end{aligned}}$$

$$\Rightarrow \boxed{\alpha + \delta = 1 + \delta \xi = 1 + \delta \ln\left(\delta/\alpha b\right)}$$

If δ is small then, $\alpha \hat{=} 1$, and putting this back into the right hand side gives improved estimate
$$\alpha \doteq 1 + \delta \ln(\delta/b) - \delta \qquad [\text{The corrections are } O(\delta^2 \ln \delta)\ ]$$

For the Earth $Q = 1370\ \mathrm{W\,m^{-2}}$, $\sigma = 5.67\times10^{-8}\ \mathrm{W\,m^{-2}\,K^{-4}}$, $T_0 = 273\ \mathrm{K}$.

So $\boxed{\alpha = \left(\frac{Q}{4\sigma T_0^4}\right)^{\frac{1}{4}} \approx 1.02119.}$

$R = 8.3\ \mathrm{J\,mol^{-1}\,K^{-1}}$, $M_v = 18\times10^{-3}\ \mathrm{kg\,mol^{-1}}$, $L = 2.5\times10^{6}\ \mathrm{J\,kg^{-1}}$ $\Rightarrow$ $a \approx 19.9$.

$\Rightarrow$ $\delta \approx 0.2$ (using $c = 0.25$).

So $\boxed{\alpha_c \approx 1.05.}$

So $\alpha < \alpha_c$, suggesting runaway greenhouse effect does not occur.

For Venus, Q is twice as large so α is increased by a factor of $2^{1/4}$ $\Rightarrow$ $\boxed{\alpha \approx 1.21}$

This is larger than α_c, so runaway greenhouse effect does occur. (ie the saturation vapour pressure is never reached).

If solar radiation were 30% smaller when the atmospheres were forming, this would not make a difference, since decreasing α by $(0.7)^{1/4} \approx 0.91$ does not change the conclusion that $\alpha < \alpha_c$ for Earth and $\alpha > \alpha_c$ for Venus.

3. Lapse rates $\quad \rho_a c_p \frac{dT}{dz} - \frac{dp}{dz} + \rho_a L \frac{dm}{dz} = 0 \quad (1) \qquad \frac{dp}{dz} = -\rho_a g \quad (2) \qquad m = \frac{\rho_v}{\rho_a} \quad (3) \qquad p = \frac{\rho_a R T}{M_a} \quad (4) \qquad p_v = \frac{\rho_v R T}{M_v} \quad (5)$

Clausius-Clapeyron eqn $\quad \frac{dp_{sv}}{dT} = \frac{\rho_v L}{T} = \frac{p_{sv}}{T^2}\frac{M_v L}{R} \quad (6)$ (using (5) under saturated conditions).

$$\Rightarrow \ln \frac{p_{sv}}{p_{sv0}} = \frac{M_v L}{R}\left(\frac{1}{T_0} - \frac{1}{T}\right) \approx \frac{M_v L}{R T_0}\left(\frac{T - T_0}{T_0}\right) \quad (*).$$

(i) dry adiabat. $\frac{dm}{dz} = 0$. So (1) $\Rightarrow \rho_a c_p \frac{dT}{dz} = \frac{dp}{dz} = -\rho_a g$ (using (2))

$$\Rightarrow \boxed{\frac{dT}{dz} = -\frac{g}{c_p}} = \Gamma_d$$

(ii) moist adiabat. $p_v = p_{sv}$. So $\quad m = \frac{\rho_v}{\rho_a} = \frac{M_v}{M_a}\frac{p_{sv}}{p}$ (using (4),(5))

$$\Rightarrow \frac{dm}{dz} = \frac{M_v}{M_a}\frac{d}{dz}\left(\frac{p_{sv}}{p}\right) = \frac{M_v}{M_a}\left[\frac{1}{p}\left(\frac{\rho_v L}{T}\right)\frac{dT}{dz} - \frac{p_{sv}}{p^2}(-\rho_a g)\right] \quad \text{(using (6),(2))}$$

$$= \frac{M_v}{M_a}\frac{\rho_v L}{p}\frac{1}{T}\frac{dT}{dz} + \frac{\rho_v g}{p} \quad \text{(using (5))}.$$

So (1) $\Rightarrow \rho_a c_p \frac{dT}{dz} + \rho_a g + \frac{\rho_a L}{T}\frac{M_v}{M_a}\frac{\rho_v L}{p}\frac{dT}{dz} + \rho_a L \frac{\rho_v g}{p} = 0$

$$\Rightarrow \boxed{\frac{dT}{dz} = -\frac{g}{c_p}\left(1 + \frac{\rho_v L}{p}\right)\Big/\left(1 + \frac{\rho_v L}{p}\frac{M_v}{M_a}\frac{L}{c_p T}\right)} = \Gamma_m$$

If $RH < 1$, we must be in the 'dry' case, since $p_v < p_{sv}$, so $T = T_0 - \frac{g}{c_p} z$.

$p_{sv}(T)$ is given approximately by the saturation curve (*), so

$$\boxed{p_{sv} \approx p_{sv0} \exp\left[-\frac{M_v L}{R T_0^2}\frac{g}{c_p} z\right]}.$$

Since m is constant, $\frac{p_v}{p} = \frac{M_a}{M_v}\frac{\rho_v}{\rho_a} = \frac{M_a}{M_v} m$ (from (4),(5)) implies p_v is proportional to p.

Moreover $\frac{dp}{dz} = -\rho_a g \approx -\frac{M_a g}{R T_0} p \Rightarrow p \approx p_0 e^{-\frac{M_a g}{R T_0} z} \Rightarrow \boxed{p_v = p_{v0} \exp\left[-\frac{M_a g}{R T_0} z\right]}$

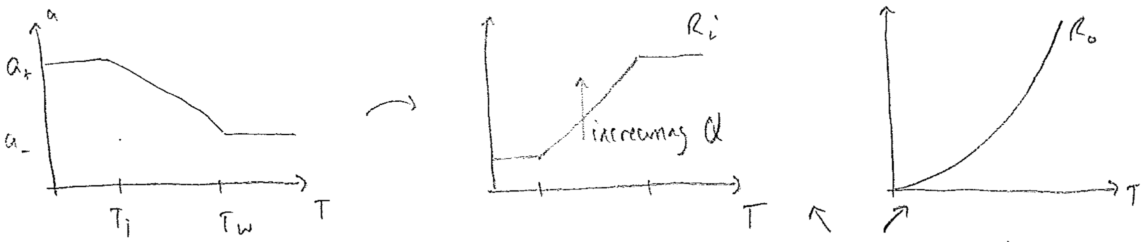
scale height

using (4), approximating $T \approx T_0$.

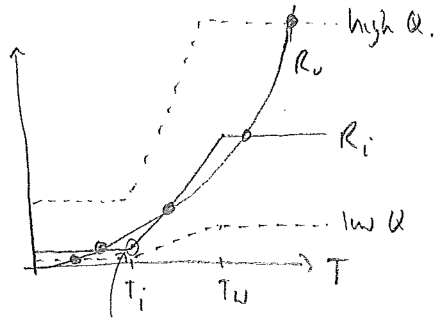
Combining, $\underbrace{\frac{p_v}{p_{sv}}}_{RH} = \underbrace{\frac{p_{v0}}{p_{sv0}}}_{RH_0} \exp\left[\frac{M_a g}{R T_0}\left(\frac{M_v}{M_a}\frac{L}{c_p T_0} - 1\right) z\right]$.

Clouds form where $RH = 1 \Rightarrow \boxed{z = \frac{\ln 1/RH_0}{\frac{M_a g}{R T_0}\left(\frac{M_v}{M_a}\frac{L}{c_p T_0} - 1\right)}}$ $\qquad$ (eg. for $RH_0 = 0.5$, $z \approx 1.2$ km)

4. Ice-albedo feedback. $c\frac{dT}{dt} = R_i - R_o$. $R_i = \frac{1}{4}Q(1-a)$ $R_o = \sigma\gamma T^4$



Steady states are intersection of these

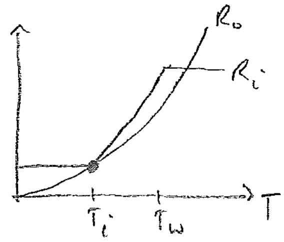


From the graph it is clear that there can be multiple intersections for intermediate values of Q, provided the slope of the central section of the R_i curve is sufficiently steep.

In particular, multiple intersections require $R_i(T_i) < R_o(T_i)$, and the slope $R_i'(T_i) = \frac{1}{4}Q\frac{a_+-a_-}{T_w-T_i}$ must be larger than $R_o'(T_i) = 4\sigma\gamma T_i^3$. The largest value that $R_i'(T_i)$ takes, while this point remains below the $R_o(T_i)$ curve is when $R_i(T_i) = R_o(T_i) \Rightarrow \frac{1}{4}Q(1-a_+) = \sigma\gamma T_i^4$

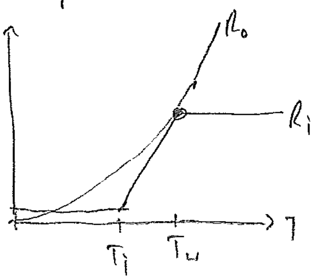
$\Rightarrow Q = \frac{4\sigma\gamma T_i^4}{1-a_+}$, so we require $\frac{\sigma\gamma T_i^4}{1-a_+}\frac{a_+-a_-}{T_w-T_i} > 4\sigma\gamma T_i^3$

$$\iff \boxed{\frac{T_w-T_i}{T_i} < \frac{a_+-a_-}{4(1-a_+)}}$$

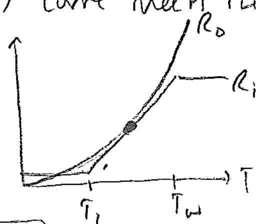


(equality would occur when the two slopes are equal)

If this condition on the slopes hold, then for smaller Q than this value ($\boxed{Q_+ = \frac{4\sigma\gamma T_i^4}{1-a_+}}$) there will clearly be multiple intersections (as in diagram above). As Q is reduced, the multiple intersections cease to occur either when $R_i(T_w)$ drops below $R_o(T_w)$:



or when the $R_i(T)$ curve meets the $R_o(T)$ curve tangentially:



In the first case, the lower bound is $\boxed{Q_- = \frac{4\sigma\gamma T_w^4}{1-a_-}}$ and this applies if $R_i'(T_w) > R_o'(T_w)$ then

i.e. $\frac{\sigma\gamma T_w^4}{1-a_-}\frac{a_+-a_-}{T_w-T_i} > 4\sigma\gamma T_w^3$

$$\iff \boxed{\frac{T_w-T_i}{T_w} < \frac{a_+-a_-}{4(1-a_-)}}$$

In the second case, we must find the value of Q for which the curves meet tangentially.

This happens when
$$\left.\begin{aligned} \tfrac{1}{4}Q(1-a) &= \sigma\gamma T^4. \\ \tfrac{1}{4}Q\,\frac{a_+-a_-}{T_w-T_i} &= 4\sigma\gamma T^3 \end{aligned}\right\} \text{ solve for } T \text{ and } Q.$$

$$1-a = 1-a_+ + \frac{a_+-a_-}{T_w-T_i}(T-T_i)$$

Write $\lambda = \dfrac{a_+-a_-}{T_w-T_i}$, then
$$\left.\begin{aligned} \tfrac{1}{4}Q(1-a_+ + \lambda(T-T_i)) &= \sigma\gamma T^4 \\ \tfrac{1}{4}Q\lambda &= 4\sigma\gamma T^3 \end{aligned}\right\} \quad 1-a_+ + \lambda(T-T_i) = \frac{\lambda T}{4}.$$

$$\Rightarrow \frac{3}{4}T = T_i - \frac{(1-a_+)}{\lambda}.$$

$$\Rightarrow T = \frac{4}{3}T_i - \frac{4}{3}\frac{(1-a_+)}{\lambda} = \frac{4}{3}\left[\frac{(1-a_-)T_i - (1-a}{a_+-a_-}\right.$$

Then
$$Q = \frac{16\sigma\gamma T^3}{\lambda} = \frac{16\sigma\gamma T^3(T_w-T_i)}{a_+-a_-} = \frac{64\cdot 16\sigma\gamma}{27}\frac{(T_w-T_i)}{(a_+-a_-)^4}\left[(1-a_-)T_i - (1-a_+)T_w\right]^3.$$

$$\Rightarrow \boxed{Q_- = \frac{512}{27}\sigma\gamma\,\frac{(T_w-T_i)\left[(1-a_-)T_i - (1-a_+)T_w\right]^3}{(a_+-a_-)^4}.}$$

(This can be seen to give the same value as in the first case if $\dfrac{T_w-T_i}{T_w} = \dfrac{a_+-a_-}{4(1-a_-)}$)