

3. Asymptotic Approximations

3.1 Definitions

- A series $\sum_{n=0}^{\infty} f_n(z)$ converges at fixed z if $\forall \varepsilon > 0, \exists N_0(z, \varepsilon) \in \mathbb{N}$

such that

$$\left| \sum_{n=m}^{\infty} f_n(z) \right| < \varepsilon \quad \forall N \geq m > N_0.$$

- A series $\sum_{n=0}^{\infty} f_n(z)$ converges to $f(z)$ at fixed z if $\forall \varepsilon > 0, \exists N_0(z, \varepsilon) \in \mathbb{N}$

such that

$$\left| f(z) - \sum_{n=0}^N f_n(z) \right| < \varepsilon \quad \forall N \geq N_0.$$

- A series converges if its terms decay sufficiently rapidly as $n \rightarrow \infty$
- Less useful in practice than might be believed.

Example $\operatorname{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt, \quad z \in \mathbb{C}.$

e^{-t^2} is a holomorphic function of $t \in \mathbb{C}$.

Thus it has a convergent power series with infinite radius of convergence

$$\sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!}$$

Integrate term by term

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)n!} = \frac{2}{\sqrt{\pi}} \left(z - \frac{z^3}{3} + \frac{z^5}{10} \dots \right)$$

Has infinite radius of convergence.

For accuracy of 10^{-5} ,

16 terms needed for $z = 2$

31 terms needed for $z = 3$

75 terms needed for $z = 5$

Holomorphic - differentiable
Analytic - convergent power series

Equivalence for functions of a complex variable - major theorem

Cancellation required between large powers... need lot of terms for good approximation

Alternative approach to approximating erf(z).

Rewrite $\text{erf}(z) = 1 - \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2} dt$

Parts $\int_z^\infty e^{-t^2} dt = \int_z^\infty \left(\frac{-1}{2t} \right) (-2te^{-t^2}) dt$ $\begin{matrix} -\frac{1}{2}t & -2te^{-t^2} \\ \frac{1}{2}t^2 & e^{-t^2} \end{matrix}$

$$= \left[-\frac{1}{2}t e^{-t^2} \right]_z^\infty - \int_z^\infty \frac{1}{2t^2} e^{-t^2} dt$$
$$= \frac{1}{2z} e^{-z^2} - \int_z^\infty \frac{e^{-t^2}}{2t^2} dt$$

Continuing the integration by parts

$$\text{erf}(z) = 1 - \frac{e^{-z^2}}{z\sqrt{\pi}} \left(1 - \frac{1}{2z^2} + \frac{1 \cdot 3}{(2z^2)^3} - \frac{1 \cdot 3 \cdot 5}{(2z^2)^5} + \dots \right)$$

This series diverges $\forall z \in \mathbb{C}$, but truncated series very useful.

3.4.

- For accuracy of 10^{-5} only two terms are needed for $z = 3$.
- Importantly The leading term is almost correct and each additional term gets us closer to the answer, with each additional correction of decreasing size until eventually they start increasing.
- This is an asymptotic series

Asymptoticness

- A sequence $\{f_n(\varepsilon)\}_{n \in \mathbb{N}_0}$ is asymptotic if $\forall n \geq 1$

$$\frac{f_n(\varepsilon)}{f_{n-1}(\varepsilon)} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0$$

• A series $\sum_{n=0}^{\infty} f_n(\varepsilon)$ is an asymptotic expansion of a function

$$f(\varepsilon) \text{ as } \varepsilon \rightarrow 0 \text{ if } \forall N \in \mathbb{N}_0 \quad \frac{f(\varepsilon) - \sum_{n=0}^N f_n(\varepsilon)}{f_N(\varepsilon)} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

• In other words, the remainder is smaller than the last term included once ε is sufficiently small.

• We write $f(\varepsilon) \sim \sum_{n=0}^{\infty} f_n(\varepsilon)$ as $\varepsilon \rightarrow 0$

• Usually first few terms are sufficient for a good approximation

• Often $f_n(\varepsilon) = a_n \varepsilon^n$ with a_n real, in which case

$$f(\varepsilon) \sim \sum_{n=0}^{\infty} a_n \varepsilon^n \text{ as } \varepsilon \rightarrow 0$$

is called an asymptotic power series.

$\left. \begin{array}{l} f_n = a_n \delta_n(\varepsilon) \\ \text{with } \{\delta_n(\varepsilon)\}_{n \in \mathbb{N}_0} \\ \text{asymptotic also common} \end{array} \right\}$

Order Notation, Big Oh

invariably $\varepsilon_0 = 0$

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• $f(\varepsilon) = O(g(\varepsilon))$ as $\varepsilon \rightarrow \varepsilon_0$ means

$$\exists K, \delta > 0 \quad \text{s.t.} \quad |f(\varepsilon)| < K |g(\varepsilon)| \quad \forall |\varepsilon - \varepsilon_0| < \delta$$

• $f(\varepsilon) = o(g(\varepsilon))$ as $\varepsilon \rightarrow \varepsilon_0$ means $\frac{f(\varepsilon)}{g(\varepsilon)} \rightarrow 0$ as $\varepsilon \rightarrow \varepsilon_0$

little
oh.

• $f(\varepsilon) = \text{ord}(g(\varepsilon))$ as $\varepsilon \rightarrow \varepsilon_0$ means

$$\exists K \in \mathbb{R} \setminus \{0\} \quad \text{s.t.} \quad \frac{f(\varepsilon)}{g(\varepsilon)} \rightarrow K \quad \text{as } \varepsilon \rightarrow \varepsilon_0$$

[Common to write
O instead of ord
in many textbooks]

Examples

$$\sin(x) = O(1), o(1), O(x), \text{ord}(x) \quad \text{as } x \rightarrow 0$$

$$\sin(x) = O(1) \quad \text{as } x \rightarrow \infty \quad \text{with suitable change in definition for limit at } \infty$$

$$\log(x) = o(x^{-\delta}) \quad \text{as } x \rightarrow 0 \quad \text{for any } \delta > 0.$$

3.2 Uniqueness and manipulation of an asymptotic series

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- If a function $f(\varepsilon) \sim \sum_{n=0}^{\infty} a_n \delta_n(\varepsilon)$ as $\varepsilon \rightarrow 0$ then induction implies that

$$\{a_n\}_{n \in \mathbb{N}_0} \text{ is uniquely determined by } a_k = \lim_{\varepsilon \rightarrow 0} \left[\frac{f(\varepsilon) - \sum_{n=0}^{k-1} a_n \delta_n(\varepsilon)}{\delta_k(\varepsilon)} \right]$$

- Uniqueness is for a given sequence $\{\delta_n(\varepsilon)\}_{n \in \mathbb{N}_0}$

- The sequence need not be unique e.g.

$$\tan \varepsilon \sim \varepsilon + \frac{\varepsilon^3}{3} + \frac{2\varepsilon^5}{15} + \dots \quad \text{as } \varepsilon \rightarrow 0$$

$$\tan \varepsilon \sim \sin \varepsilon + \frac{1}{2} (\sin \varepsilon)^3 + \frac{3}{8} (\sin \varepsilon)^5 + \dots \quad \text{as } \varepsilon \rightarrow 0$$

- Uniqueness for a given function... two functions may share the same asymptotic expansion e.g.

$$e^\varepsilon \sim \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \quad \text{as } \varepsilon \rightarrow 0$$

$$e^\varepsilon + e^{-1/\varepsilon^2} \sim \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \quad \text{as } \varepsilon \rightarrow 0$$

- Two distinct functions with the same asymptotic power series can only differ by a function that is not holomorphic as two holomorphic functions with the same power series are identical.
- Asymptotic expansions can be naively added, subtracted, divided, multiplied and divided (though the sequence eg. the $\{\delta_n(\epsilon)\}_{n \in \mathbb{N}_0}$ may be larger).
- This underlies expansion method for algebraic equations.
- One series can be substituted into another, but take care with exponentials ... always expand exponents to $\text{ord}(1)$.

Example $f(z) = e^{z^2}$ $z = \frac{1}{\epsilon} + \epsilon$ Naively $f(z) \sim e^{\frac{1}{2}\epsilon^2}$ at leading order X

$$f(z) = \exp\left(\left(\frac{1}{\epsilon} + \epsilon\right)^2\right) = \exp\left(\frac{1}{\epsilon^2} + 2 + \epsilon^2\right) = e^{\frac{1}{2}\epsilon^2} \cdot e^2 \cdot \left(1 + \epsilon^2 + \frac{(\epsilon^2)^2}{2!} + \frac{(\epsilon^2)^3}{3!} + \dots\right)$$

$\equiv \leftarrow$ Naive answer wrong by e^2 .

- Sine and Cosine and complex exponentials require analogous care in this context.
- Asymptotic expansions can be integrated term by term with respect to ε resulting in the correct asymptotic expansion for the integral.
- In general asymptotic expansions cannot be differentiated safely

Example

$$f(\varepsilon) = \varepsilon \cos\left(\frac{1}{\varepsilon}\right) = o(\varepsilon) \text{ as } \varepsilon \rightarrow 0$$

$$f'(\varepsilon) = \frac{1}{\varepsilon} \sin\left(\frac{1}{\varepsilon}\right) + \cos\left(\frac{1}{\varepsilon}\right) = o\left(\frac{1}{\varepsilon}\right) \text{ as } \varepsilon \rightarrow 0$$

Differentiating the asymptotic expansion with the $o(\varepsilon)$ start would naively give $o(1)$... but the derivative is $o\left(\frac{1}{\varepsilon}\right)$ as $\varepsilon \rightarrow 0$.

- Terms move down an asymptotic expansion with differentiation (eg. $\frac{d}{dx} x^n = nx^{n-1}$) and thus terms at higher orders may cause problems on differentiation.

- Often, first few terms sufficient. If higher accuracy required, ...
optimal truncation : truncate asymptotic series at smallest term

3.4 Parametric Expansions

- Integrals, differential equations and partial differential equations involve functions with one, or more, variables $f(x, \varepsilon)$ with ε a small parameter.

- There is an obvious generalisation of the definition of an asymptotic expansion by allowing the coefficients to depend on x . For fixed x

$$f(x; \varepsilon) \sim \sum_{n=0}^{\infty} a_n(x) \delta_n(\varepsilon) \quad \text{as } \varepsilon \rightarrow 0 \quad \text{if and only if}$$

$$\frac{1}{\delta_N(\varepsilon)} \left[f(x; \varepsilon) - \sum_{n=0}^N a_n(x) \delta_n(\varepsilon) \right] \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0$$