Modular Forms

Problem Sheet 2

HT 2018

1. For any $\Gamma \le \mathrm{SL}_2(\mathbb{Z})$ of finite index show that the modular curve X_Γ has only finitely many cusps. Show that the width of each cusp is finite.

2. Compute the indices $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_1(N)]$ and $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]$, and thus also $[\mathrm{PSL}_2(\mathbb{Z}) : \overline{\Gamma_1(N)}]$ and $[\mathrm{PSL}_2(\mathbb{Z}) : \overline{\Gamma_0(N)}]]$.

   [*Hint: Note* $\Gamma(N) \trianglelefteq \Gamma_1(N) \trianglelefteq \Gamma_0(N) \le \mathrm{SL}_2(\mathbb{Z})$ *and compute* $[\Gamma_1(N) : \Gamma(N)]$ *by constructing a suitable homomorphism* $\Gamma_1(N) \to \mathbb{Z}/(N)$ *etc.*]

3. Let p be prime.

   (a) Show that the cusps for the congruence subgroup $\Gamma_0(p)$ are the classes of 0 and ∞. Find the width of each cusp and a generator for its stabiliser.

   (b) Prove that (the linear fractional transformations attached to the matrices)

   $$\left\{ \begin{pmatrix} 1 & 0 \\ kp & 1 \end{pmatrix} : 0 \le k \le p-1 \right\}$$

   is a complete set of coset representatives for $\overline{\Gamma_0(p^2)}$ in $\overline{\Gamma_0(p)}$. Show that $\Gamma_0(p^2)$ has $p+1$ cusps: the classes of $0, \infty$ and $1/kp$ for $k = 1, \cdots, p-1$.

4. Let $X_0(3) := \mathfrak{H}^\star/\Gamma_0(3)$ be the compact Riemann surface associated to the congruence subgroup $\Gamma_0(3)$. Draw a fundamental domain $D_{\Gamma_0(3)}$ for $\Gamma_0(3)$, and define explicit maps giving the local coordinate around each cusp and elliptic point. Draw a triangulation of $D_{\Gamma_0(3)}$, and hence by identifying appropriate edges one for $X_0(3)$. From your triangulation of $X_0(3)$ compute its genus.

5. Write $X_0(N)$ and $X_1(N)$ for the compact Riemann surfaces associated to the groups $\Gamma_0(N)$ and $\Gamma_1(N)$, respectively.

   (a) Prove that (the linear fractional transformations attached to the matrices)

   $$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -2 & 1 \\ -5 & 2 \end{pmatrix} \right\}$$

   is a complete set of coset representatives for $\overline{\Gamma_1(5)}$ in $\overline{\Gamma_0(5)}$. Show that $X_1(5)$ has 4 cusps: the orbits of $0, \frac{2}{5}, \frac{1}{2}, \infty$. For each cusp find the width of the cusp and a generator for its stabiliser.

(b) Show that $X_1(5)$ has genus zero.

[*Recall in Sheet 1 we already showed that* $X_1(5)$ *has no elliptic points.*]

(c) Show that $\Gamma_0(5)$ has no elliptic points of order 3, and 2 elliptic points of order 2.

[*Hint: Consider coset representatives for* $\Gamma_0(5)$ *in* $\mathrm{PSL}_2(\mathbb{Z})$.]

(d) Prove that $X_0(5)$ has genus zero.

[*More generally* $X_0(p)$ *has genus* $(p-5)/12$ *when* $p \equiv 5 \bmod 12$. *Can you see why?*]