Quantum Field Theory

Warm-up exercises

1. Consider the classical Lagrangian of a particle of mass m and charge q, moving in an electromagnetic field

$$\mathcal{L} = \frac{m}{2}(\partial_t x^i)^2 + qA_i(x)\partial_t x^i - q\phi(x)\,, \tag{1}$$

where $\phi(x)$ is the electric potential and $A_i(x)$ is the magnetic potential.

   (a) Find the canonical momentum conjugate to the coordinate x_i.

   (b) Derive the equations of motion corresponding to this Lagrangian.

   (c) Find the Hamiltonian of the system.

Compare your results to a free particle case.

2. Consider the Hamiltonian of a quantum harmonic oscillator:

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}\,. \tag{2}$$

   (a) Diagonalise the Hamiltonian by introducing ladder operators.

   (b) Calculate the expectation values of the number operator $N \sim a^\dagger a$. Calculate the expectation values of the x and p operators for a general state $|n\rangle$.

   (c) Evaluate the variances Δx, Δp and ΔN in the state $|n\rangle$ and use them to determine the Heisenberg uncertainty of $|n\rangle$.

   (d) Show that the coherent state

$$|\alpha\rangle = e^{\alpha p}|0\rangle \tag{3}$$

   is an eigenstate of the annihilation operator.

   (e) Calculate the time-dependent expectation values of x, p and N in the coherent state: $\langle\alpha|x(t)|\alpha\rangle$, $\langle\alpha|p(t)|\alpha\rangle$ and $\langle\alpha|N(t)|\alpha\rangle$. Find the corresponding variances to determine the uncertainty of the state $|\alpha\rangle$.

3. Consider the following action for a relativistic point particle

$$S = -\alpha \int ds = -\alpha \int \sqrt{\eta_{\mu\nu} \frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\nu}{\partial \tau}} d\tau \,, \tag{4}$$

where τ parametrises the worldline of the particle $X^\mu(\tau)$.

(a) Show that the action is invariant under Poincaré transformations.

(b) Show that the action is invariant under reparametrisation of the worldline time $\tau \to \tau'(\tau)$.

(c) Show that

$$p^\mu = -\frac{\alpha \dot{X}^\mu}{\sqrt{\eta_{\nu\rho} \dot{X}^\nu \dot{X}^\rho}} \,. \tag{5}$$

(d) Parametrise the path by the time coordinate $t = x^0$ and take the non-relativistic limit $|\partial_0 x^i| \ll 1$ to determine the value of the constant α.