Calculus of Variations - Problem Sheet 1

Trinity Term 2019

1. Find the extremals of the functionals (assume that y is prescribed at $x = a$ and $x = b$):

(a) $\displaystyle\int_a^b (y^2 - y'^2 - 2y \; cos \; 2x)dx$

(b) $\displaystyle\int_a^b \frac{y'^2}{x^3} dx$

(c) $\displaystyle\int_a^b (y^2 + y'^2 - 2ye^x)dx$

2. Find the extremals of

(a) $\displaystyle\int_0^1 (y^2 + y' + y'^2)dx$ subject to $y(0) = 0, y(1) = 1$

(b) $\displaystyle\int_0^1 \frac{y'^2}{x^3} dx$ subject to $y(0) = 1, y(1) = 2$

(c) $\displaystyle\int_0^1 (y'^2)dx + \{y(1)\}^2$ subject to $y(0) = 1$.

3. Show that the problem of finding extremals of

$$J[y] = \int_a^b F(x, y, y')dx,$$

among all twice continuously differentiable functions y for which $y(a)$ is prescribed, leads to the Euler equation

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{\partial F}{\partial y}$$

and to the natural boundary condition

$$\frac{\partial F}{\partial y'}|_{x=b} = 0.$$

Find the extremal of $\displaystyle\int_0^1 \left(\frac{1}{2}y'^2 + yy' + y' + y\right) dx$ among all y with $y(0) = 1$.

4. Show that the Euler equation of the functional

$$\int_{x_0}^{x_1} F(x, y, y', y'')dx$$

has the first integral $F_{y'} - \dfrac{d}{dx}F_{y''} =$ constant if $F_y \equiv 0$ and the first integral $F - y'(F_{y'} - \frac{d}{dx}F_{y''}) - y''F_{y''} = \text{constant}$ if $F_x \equiv 0$.

5. Find the extremals of the functional

$$\int_0^{\frac{\pi}{2}} (y'^2 + z'^2 + 2yz)dx$$

subject to $y(0) = 0, y\left(\frac{\pi}{2}\right) = 1, z(0) = 0, z\left(\frac{\pi}{2}\right) = 1$.