ALGEBRA III – GROUP THEORY – SHEET 1

Isomorphism Theorems. Free Groups. Generators and Relations. Composition Series.

1. Let H and K be subgroups of a group G. Show that $HK = \{hk : h \in H, k \in K\}$ is a subgroup of G if and only if

$$HK = KH.$$

2. Let $K \lhd G$. Denote $\bar{G} = G/K$ and let $\bar{H} \leqslant \bar{G}$. Show that

$$H = \left\{h \in G : hK \in \bar{H}\right\}$$

is a subgroup of G, containing K as a normal subgroup and such that $H/K = \bar{H}$. Show further that if $\bar{H}$ is normal in $\bar{G}$ then H is normal in G.

3. Identify the following groups from the given presentations.

(i) $G_1 = \langle x \mid x^6 = e\rangle$.
(ii) $G_2 = \langle x, y \mid xy = yx\rangle$.
(iii) $G_3 = \langle x, y \mid x^3y = y^2x^2 = x^2y\rangle$.
(iv) $G_4 = \langle x, y \mid xy = yx,\ x^5 = y^3\rangle$.
(v) $G_5 = \langle x, y \mid xy = yx,\ x^4 = y^2\rangle$.

[For G_4 you may wish to consider the homomorphism $\phi\colon \mathbb{Z}^2 \to \mathbb{Z}$ given by $\phi(a, b) = 3a + 5b$.]

4. Let G be a non-abelian group of order 8. We know then that all elements have order 1, 2 or 4 and there exist elements of order 4. [*How do we know this*?] Let a be an element of order 4, set $A = \langle a\rangle$ and let $b \in G \setminus A$. Show that $b^{-1}ab = a^{-1}$ and that either $b^2 = e$ or $b^2 = a^2$. Use this to prove that up to isomorphism there are just five groups of order 8. [*You may assume that any finite abelian group is a direct product of cyclic groups.*]

5. Let $G = \langle x, y \mid x^2 = e = y^2\rangle$ and let D_∞ denote the isometry group of $\mathbb{Z}$, the *infinite dihedral group*.

(i) Show that G is infinite.

(ii) Let $z = xy$. Show that every element of G can be uniquely written as z^k or yz^k where k is an integer. Show that $G = \langle y, z \mid y^2 = e,\ yzy = z^{-1}\rangle$.

(iii) Show that $y(n) = -n$ and $z(n) = n + 1$ are elements of D_∞. Deduce that $G \cong D_\infty$.

6. (i) Let $n \geqslant 1$. Show that (12) and $(123\ldots n)$ generate S_n.

(ii) Show that $\mathbb{Q}$ is not finitely generated.

7. Write down all possible composition series of the following groups, verifying the Jordan-Hölder Theorem where appropriate.

$$\mathbb{Z}_{12}, \qquad D_{10}, \qquad D_8, \qquad Q_8.$$

8. (Optional) Let

$$\{e\} \lhd G_1 \lhd G \qquad \text{and} \qquad \{e\} = H_0 \lhd H_1 \lhd \cdots \lhd H_r = G$$

be two composition series for a group G. Why is $r \geqslant 2$? Why is $r = 2$ if $H_{r-1} = G_1$? Show that if $H_{r-1} \neq G_1$ then $G_1 \cap H_{r-1} = \{e\}$. Show that G_1H_{r-1} is normal in G and that $G/G_1 = H_{r-1}$. Deduce that $r = 2$.