ALGEBRA III – GROUP THEORY – SHEET 2

Solvable Groups. Semi-direct Products and Extensions. Sylow's Theorems.

1. Let A_∞ denote the even permutations of $\mathbb{N}$, thought of as

$$A_\infty = \bigcup_{n=1}^{\infty} A_n.$$

Show that A_∞ is an infinite simple group.

2. Let G be a group and G' denote its derived subgroup. We showed in lectures that $G' \lhd G$.

(i) Show that if $H \lhd G$ and G/H is Abelian then $G' \leqslant H$.

(ii) Conversely, show that if $G' \leqslant H \leqslant G$ then $H \lhd G$ and G/H is Abelian.

3. Given two groups N, H and a homomorphism $\varphi \colon H \to \mathrm{Aut}(N)$, verify that the semi-direct product $N \rtimes_\varphi H$ does indeed satisfy the group axioms.

4. Verify directly Sylow's three theorems for the following groups:

$$S_3, \qquad D_{12}, \qquad A_4, \qquad S_4.$$

5. Let P be a non-trivial group of order p^m, where p is prime and $m > 0$.

By considering the conjugation action of P on itself prove that there is a non-identity element z such that $xz = zx$ for all $x \in P$.

Show that $K = \langle z \rangle$ is a normal subgroup of P.

Deduce, by induction on m, or otherwise, that finite groups of prime power order are solvable.

6. Show that a group of order 1694 is solvable.

7. Let G be a group of order 30.

(i) Explain why one of the following holds:

- There is a normal subgroup N of order 5 and a subgroup H of order 3;
- There is a normal subgroup N of order 3 and a subgroup H of order 5.

Deduce that G has a cyclic normal subgroup K of order 15.

(ii) Let y be a generator of K and x be an order 2 element. Show that

$$G = \left\{x^i y^j : 0 \leqslant i \leqslant 1,\ 0 \leqslant j \leqslant 14\right\}$$

and that $G \cong C_{15} \rtimes_\varphi C_2$ where $\varphi \colon C_2 \to \mathrm{Aut}(C_{15})$ is a homomorphism.

(iii) Let ψ be an automorphism of K such that $\psi(\psi(y)) = y$. Show that $\psi(y) = y$ or y^4 or y^{11} or y^{14}.

(iv) Deduce that there are (up to isomorphism) at most four groups of order 30. Show that there are precisely four by exhibiting four non-isomorphic groups of order 30.