Electromagnetism

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Chapters

Reading list

R. Feynman, Lectures in Physics, Vol.2. Electromagnetism, Addison Wesley.

J.D. Jackson, Classical Electrodynamics, John Wiley.

Landau and Lifshitz, The Classical Theory of Fields (to read if you are brave enough!)

1 Introduction to Electrostatics

This is the subject of static electricity, which we've all played with: recall rubbing a pen on your cat and picking up bits of paper. We interpret this as charging up or adding charge to the pen. As greeks didn't have pens, they used amber. The greek word for amber is electron and this is where the subject gets its name from.

1.1 Coulomb's law

We will assume the following facts about charge:

- there exist charges in nature;
- charge can be positive or negative;
- charge cannot be created or destroyed;

For a more sophisticated standpoint, we may know that charge is discrete, and that electrons have charge -1 and protons charge +1 in suitable units, but for our purposes we want to think of charge as continuous. As a mathematical idealization, we represent charge by a real function, the *charge density* $\rho(x, y, z, t)$, with the property that the total charge Q in a region V of space is

$$Q = \int_V \rho dV = \iiint_V \rho \; dxdydz. \tag{1}$$

Other idealizations are possible, and we shall sometimes use them. For example, it may be convenient to think of charges residing on a surface, something two-dimensional, and then *charge surface density* $\sigma(x, y, t)$ is charge per unit area, so that the total charge on a piece S of surface is

$$Q = \int_S \sigma dS; \tag{2}$$

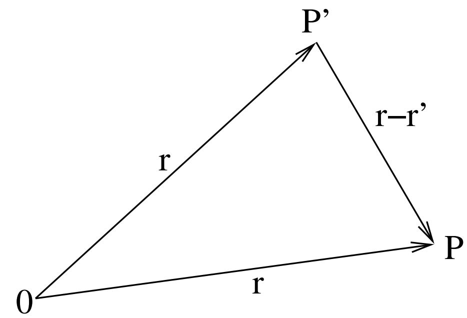
similarly, for a one-dimensional distribution of charge, *charge line density* is charge per unit length on some curve. Finally, a *point charge* q corresponds to a finite, nonzero charge localized at a point.

After introducing the concept of charge, the natural question is: how do charges interact with each other? An important experimental fact about point charges is the

Coulomb's Law: Given two point particles P, P' with charges q, q' at positions $\vec{r}, \vec{r}\,'$, the electric force on P due to P' is

$$\vec{F} = \frac{1}{4\pi\epsilon_0} qq' \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3} \tag{3}$$

where ϵ_0 is a constant and accounts for the correct units, see figure.



We treat this law as an experimental fact. To see how one might have discovered it, note that:

- it is an inverse-square law, acting along the line joining the particles;
- it is proportional to the product of the charges, so that like charges repel and unlike charges attract.

1.2 Electric field and scalar potential

It is convenient to think of Coulomb's law as follows: P' generates an *electric field* to which P responds. Put P' at the origin and define the electric field at $\vec{r}$ to be

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} q' \frac{\vec{r}}{r^3} \tag{4}$$

Then P, which has charge q, when placed in the field generated by P' is subject to (or "feels") the force

$$\vec{F} = q\vec{E} \tag{5}$$

Replace P' by various particles $P_1, P_2, ..., P_n$, at points $\vec{r}_1, \vec{r}_2, ..., \vec{r}_n$ with charges $q_1, q_2, ..., q_n$. The forces will add as vectors so if we defined the electric field now as

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{q_i(\vec{r} - \vec{r}_i)}{|\vec{r} - \vec{r}_i|^3}, \tag{6}$$

then, as before, P placed at $\vec{r}$ is subject to the force

$$\vec{F}(\vec{r}) = q\vec{E}(\vec{r}) \tag{7}$$

The electric field (6) generalizes easily to the electric field for a volume distribution of charge density

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \rho(\vec{r}\,') \frac{(\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} dx'dy'dz', \tag{8}$$

It is instructive to ask how (8) reduces to (6). In order to answer that question we need to understand which charge density $\rho(x, y, z)$ corresponds to a point charge of magnitude q located at (x_0, y_0, z_0). This function should be such that it vanishes identically outside the point (x_0, y_0, z_0) but at the same time

$$\int_V \rho(x, y, z) dV = q \tag{9}$$

for any volume including the point (x_0, y_0, z_0). We introduce the one-dimensional *Dirac delta function* $\delta(x)$, defined by the following properties:

1. $\delta(x) = 0, \qquad \text{for } x \neq 0$
2. $\int \delta(x) dx = 1, \qquad$ if the region of integration includes $x = 0$, and is zero otherwise.

Useful properties, which can be derived from the two above are the following:

$$\int f(x)\delta(x-a)dx = f(a), \qquad \text{if the region of integration includes } x = a, \tag{10}$$

$$\delta(f(x)) = \sum_i \frac{1}{|f'(x_i)|}\delta(x - x_i), \qquad \text{with } f(x_i) = 0. \tag{11}$$

In higher dimensions the Dirac delta function is just the product of the cartesian delta functions. We see that the volume charge density with the correct properties to correspond to a point charge of magnitude q located at (x_0, y_0, z_0) is given by

$$\rho(x, y, z) = q\delta(\vec{r} - \vec{r}_0) \equiv q\delta(x - x_0)\delta(y - y_0)\delta(z - z_0) \tag{12}$$

By using the properties of the delta function you can check that for $\rho(\vec{r}) = \sum q_i \delta(\vec{r} - \vec{r}_i)$ (8) reduces to the potential (6) for a collection of particles.

We want to explore (6). Note first that it is the gradient of another function:

$$\vec{E} = -\nabla\Phi, \quad \Phi = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{|\vec{r} - \vec{r}_i|}. \tag{13}$$

This scalar function Φ is called the *electric or scalar potential.* The scalar potential has a very nice physical interpretation. Let us think about Newton's equations of motion for a charged particle of mass m and charge q subject to the force (7)

$$m\vec{a} = \vec{F} = q\vec{E} = -q\nabla\Phi \tag{14}$$

Then

$$\frac{d}{dt}\left(\frac{1}{2}m\vec{v}\cdot\vec{v}\right) = m\vec{v}\cdot\vec{a} = -q\vec{v}\cdot\nabla\Phi = -q\sum_{i=1}^{3}\frac{\partial\Phi}{\partial x^i}\frac{dx^i}{dt} = -q\frac{d\Phi}{dt} \tag{15}$$

so that

$$\mathcal{E} = \frac{1}{2}m|\vec{v}|^2 + q\Phi = \text{constant} \tag{16}$$

In the motion of the charged particle, the kinetic energy changes with time, but the sum of the kinetic energy and $q\Phi$ is constant; thus $q\Phi$ is the *potential energy* of the particle of charge q in the electric field with potential Φ. [1]

1.3 Gauss law and Poisson equation

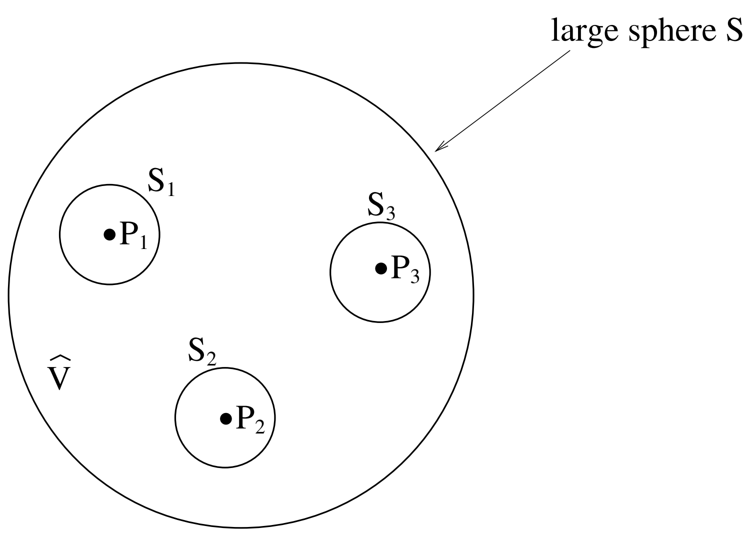
In the previous section we have seen that the electric field for a collection of particles is the gradient of a scalar function. In particular, therefore

$$\nabla \wedge \vec{E} = 0, \tag{17}$$

From (6) we can also explicitly check

$$\nabla \cdot \vec{E} = \nabla^2\Phi = 0, \qquad \text{except at } \vec{r} = \vec{r}_i \tag{18}$$

Consider the diagram



i.e. S is a large sphere containing all P_i, S_i is a small sphere containing only P_i and $\hat{V}$ is the region in between. Then $\nabla \cdot \vec{E} = 0$ on $\hat{V}$, so

[1] Note that this is completely analogue to the usual (gravitational) potential energy, the mass m is replaced by the charge q and the potential gh is replaced by the potential Φ.

$$\int_S \vec{E} \cdot d\mathbf{S} - \sum_{i=1}^{n} \int_{S_i} \vec{E} \cdot d\mathbf{S} = \int_{\hat{V}} \nabla \cdot \vec{E} dV = 0 \tag{19}$$

the first two are surface integrals (with $d\mathbf{S}$ pointing away from the center of the corresponding sphere), while the last one is a volume integral. Make sure you understand the signs in this equation. Let us focus in the integral over S_1 and consider only the charge P_1. Taking $\vec{r}_1 = 0$ for simplicity we obtain:

$$\begin{aligned}
\int_{S_1} \vec{E} \cdot d\mathbf{S} &= \int_{S_1} \frac{1}{4\pi\epsilon_0} q_1 \frac{\vec{r}}{r^3} \cdot \mathbf{n} dS \\
&= \frac{1}{4\pi\epsilon_0} q_1 \int \frac{1}{r^2} r^2 \sin\theta d\theta d\phi \\
&= \frac{1}{\epsilon_0} q_1,
\end{aligned}$$

in the second step we used $\mathbf{n} = \frac{\vec{r}}{r}$. Now, the divergence of the electric field generated by the other charges vanishes identically inside S_1, so we can use the divergence theorem to conclude that they will not contribute to the integral above. Each term in the sum (19) can be calculated like this, so we obtain:

$$\int_S \vec{E} \cdot d\mathbf{S} = \sum_{i=1}^{n} \int_{S_i} \vec{E} \cdot d\mathbf{S} = \frac{1}{\epsilon_0}(q_1 + q_2 + ... + q_n) = \frac{1}{\epsilon_0} \times \text{total charge inside S} \tag{20}$$

This is the *Gauss's Law* in its integral form:

Gauss's Law: The flux of $\vec{E}$ out of $V = \frac{1}{\epsilon_0} \times$ total charge in V.

If we have a smoothed out charge density instead of point charges, Gauss's law would read

$$\int_S \vec{E} \cdot d\mathbf{S} = \frac{1}{\epsilon_0} \int_V \rho dV, \tag{21}$$

where V is the volume inside the closed surface S. We will assume this to be true.

Now, by Divergence theorem we have

$$\int_V \left(\nabla \cdot \vec{E} - \frac{1}{\epsilon_0} \rho \right) dV = 0 \tag{22}$$

but if this is to hold for all possible regions V, then

$$\nabla \cdot \vec{E} = \frac{1}{\epsilon_0} \rho, \tag{23}$$

which is the *differential version of Gauss's Law.*

This together with $\vec{E} = -\nabla\Phi$ implies *Poisson's equation*

$$\nabla^2\Phi = -\frac{1}{\epsilon_0}\rho \tag{24}$$

What is the solution for this equation? remember that for point particles we had (see (13))

$$\Phi = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{|\vec{r}-\vec{r}_i|} \tag{25}$$

so we might guess that the solution of (24) is

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\iiint_V \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}dx'dy'dz' \tag{26}$$

where V is the region in which $\rho \neq 0$. In order check that this is the correct guess, we need to apply the Laplacian to both sides. In doing so we are led to consider

$$\nabla^2\left(\frac{1}{r}\right) = \frac{1}{r}\frac{d^2}{dr^2}\left(r.\frac{1}{r}\right) = 0, \qquad \text{for } r \neq 0 \tag{27}$$

At $r = 0$ we need to be a little bit careful. Indeed, integrating around a little sphere around the origin we find

$$\int_V \nabla^2\left(\frac{1}{r}\right)dV = \int_V \nabla\cdot\nabla\left(\frac{1}{r}\right)dV = \int_S \mathbf{n}\cdot\nabla\left(\frac{1}{r}\right)ds = -4\pi \tag{28}$$

hence we arrive to the following beautiful equation

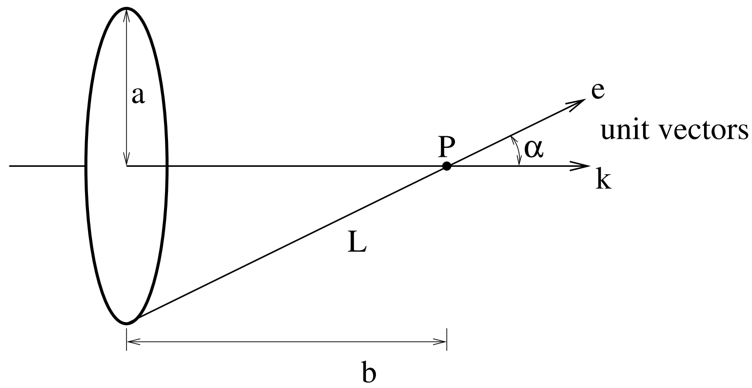
$$\nabla^2\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) = -4\pi\delta(\vec{r}-\vec{r}\,') \tag{29}$$

using this it is immediate to check that (26) is a solution to the Poisson equation. Incidentally, also note that (29) implies the Poisson equation also works for point charges, where the charge density has the form (12).

An example

The main problem of Electrostatics is to obtain $\vec{E}$ given the charge distribution. We have solved that problem with (26): Given the density ρ, integrate this to find Φ and then $\vec{E} = -\nabla\Phi$. However, in simple cases we can go straight to $\vec{E}$ by using (4). Here is an example:

Total charge Q is spread out uniformly round a plane circular wire of radius a. Find the electric field at a point P on the axis of the circle, at a distance b from the centre.



We can do this one directly from the inverse square law: $\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}}{r^3}$.

Cut the circular wire into elements, each of length $ad\theta$; then each contains charge equal to $\frac{Q}{2\pi}d\theta$ and so contributes $\frac{Q}{2\pi}d\theta\frac{1}{4\pi\epsilon_0}\frac{\vec{e}}{L^2}$, (with $\vec{e}$ as in the figure) to $\vec{E}$ at P. Adding these up around the circle leads, by symmetry, to a vector along $\vec{k}$. Note, from the diagram, that $\vec{e}\cdot\vec{k} = \cos\alpha$, so that $\vec{E}(p) = E\vec{k}$, with E a scalar and given by

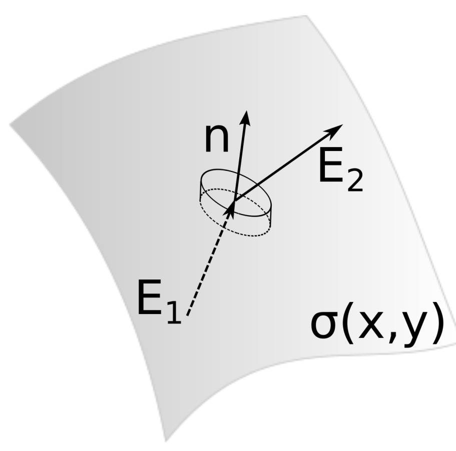
$$\begin{aligned} E &= \int \frac{Q}{2\pi}\cdot d\theta\cdot\frac{1}{4\pi\epsilon_0}\cdot\frac{\cos\alpha}{L^2} \\ &= \frac{Q}{4\pi\epsilon_0}\frac{\cos\alpha}{L^2} \\ &= \frac{Q}{4\pi\epsilon_0}\frac{b}{(a^2+b^2)^{3/2}} \end{aligned}$$

which is the answer.

It is trickier to find $\vec{E}$ for a point P' not on the axis.

Another example: Discontinuity as you cross a surface density

Suppose we want to compute the discontinuity in the electric field as we cross a surface layer of charge, with surface density $\sigma(x,y)$, see figure.



In order to answer this question we can use the Gauss law in its integral form (21) for a volume similar to a coin (as shown in the figure), with one face on each side of the surface and a negligible thickness. We obtain

$$\int_S \vec{E} \cdot d\mathbf{S} = \int_S \vec{E}_2 \cdot \vec{n} dS - \int_S \vec{E}_1 \cdot \vec{n} dS = \frac{1}{\epsilon_0} \int_V \rho dV = \frac{1}{\epsilon_0} \int_S \sigma(x,y) dS \tag{30}$$

where $\vec{n}$ is the normal to the surface. Hence we have a discontinuity in the normal component of the electric field as we cross the layer

$$\epsilon_0 \left(\vec{E}_2 - \vec{E}_1 \right) \cdot \vec{n} = \sigma(x,y) \tag{31}$$

You can now try to use $\nabla \wedge \vec{E} = 0$ together with Stokes theorem to show that the tangential components are continuous.

1.4 Boundary problems, Green's functions

If electrostatic problems always involved localized distributions of charge with no boundary surfaces, the solution would be simply given by (26) or (8) and this would be an eight hours course :-). Unfortunately, many problems involve finite regions of space with prescribed boundary conditions on the bounding surfaces. There are two natural boundary conditions:

- *Dirichlet boundary conditions*: The potential Φ is specified on a boundary surface.
- *Neumann boundary conditions*: The normal derivative of the potential $\frac{\partial \Phi}{\partial n} = \vec{n} \cdot \nabla \Phi$ is specified on a boundary surface.

To handle such problems we will make use of the *Green's theorem*:

$$\int_V \left(\phi \nabla^2 \psi - \psi \nabla^2 \phi \right) dV = \int_S \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) dS \tag{32}$$

where ϕ and ψ are two arbitrary scalar fields, S is the boundary of V and $\frac{\partial}{\partial n}$ is the normal derivative at the surface S, directed outwards from inside the volume V.

In order to exploit this theorem in the context of electrostatics we introduce the concept of *Green's function*. A Green's function is a function $G(\vec{r}, \vec{r}\,')$ which satisfies

$$\nabla'^2 G(\vec{r}, \vec{r}\,') = -4\pi \delta(\vec{r} - \vec{r}\,') \tag{33}$$

Of course, the solution to that equation is not unique, but we know the general solution is of the form:

$$G(\vec{r}, \vec{r}\,') = \frac{1}{|\vec{r} - \vec{r}\,'|} + F(\vec{r}, \vec{r}\,') \tag{34}$$

where $F(\vec{r}, \vec{r}\,')$ satisfies the Laplace's equation inside the volume V

$$\nabla'^2 F(\vec{r}, \vec{r}') = 0 \tag{35}$$

we will use the freedom to choose $F(\vec{r}, \vec{r}')$ momentarily. Next, let us consider Green's theorem with $\phi = \Phi(\vec{r}')$ the scalar potential and $\psi = G(\vec{r}, \vec{r}')$ the Green's function. We obtain

$$\int_V \left(\Phi(\vec{r}')\nabla'^2 G(\vec{r}, \vec{r}') - G(\vec{r}, \vec{r}')\nabla'^2\Phi(\vec{r}')\right) dV' = \int_S \left(\Phi(\vec{r}')\frac{\partial G(\vec{r}, \vec{r}')}{\partial n'} - G(\vec{r}, \vec{r}')\frac{\partial \Phi(\vec{r}')}{\partial n'}\right) dS' \tag{36}$$

Using (33) and the properties of the delta function we obtain

$$\int_V \Phi(\vec{r}')\nabla'^2 G(\vec{r}, \vec{r}')dV' = -4\pi\Phi(\vec{r}) \tag{37}$$

where we have assumed that the observation point $\vec{r}$ is inside the volume V. In this case we obtain

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{r}')G(\vec{r}, \vec{r}')dV' + \frac{1}{4\pi}\int_S \left(G(\vec{r}, \vec{r}')\frac{\partial \Phi(\vec{r}')}{\partial n'} - \Phi(\vec{r}')\frac{\partial G(\vec{r}, \vec{r}')}{\partial n'}\right) dS' \tag{38}$$

Now we can use the freedom in the definition of the Green's function. If we are solving a problem with Dirichlet boundary conditions we demand

$$G_D(\vec{r}, \vec{r}') = 0, \qquad \text{for } \vec{r}' \text{ in } S. \tag{39}$$

Once we have found a Green's function with these boundary conditions, the solution for the potential is simply given by

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{r}')G_D(\vec{r}, \vec{r}')dV' - \frac{1}{4\pi}\int_S \left(\Phi(\vec{r}')\frac{\partial G_D(\vec{r}, \vec{r}')}{\partial n'}\right) dS' \tag{40}$$

For Neumann boundary conditions we must be more careful. We cannot require the normal derivative of G to vanish: application of the Gauss theorem to the definition of Green's function gives

$$\int_S \frac{\partial G(\vec{r}, \vec{r}')}{\partial n} ds' = -4\pi \tag{41}$$

The easiest condition we can require is

$$\frac{\partial G_N(\vec{r}, \vec{r}')}{\partial n} = -\frac{4\pi}{A} \tag{42}$$

where A is the total surface area of S. Then the solution is

$$\Phi(\vec{r}) = \langle\Phi\rangle_S + \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{r}')G_N(\vec{r}, \vec{r}')dV' + \frac{1}{4\pi}\int_S G_N(\vec{r}, \vec{r}')\frac{\partial \Phi(\vec{r}')}{\partial n'}dS' \tag{43}$$

where $\langle \Phi \rangle_S$ is the average value of the potential over the whole surface. In the case of an "exterior problem" the volume is bounded by two surfaces, one finite and closed and the other one at infinity. In this case the surface area diverges and $\langle \Phi \rangle_S$ vanishes.

Let us add that for both boundary conditions we can always choose the Green's function to be symmetric under the interchange of $\vec{r}$ and $\vec{r}\,'$.

In practice it can be quite hard to find the Green's function with the appropriate boundary conditions. In the next section we will see some examples, and introduce different methods to solve boundary problems in electrostatics.

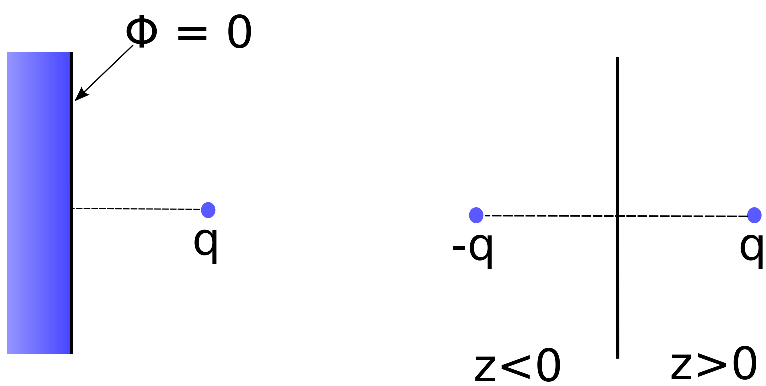
2 Boundary-value problems in electrostatics

Many problems in electrostatics involve boundary surfaces on which either the potential or the normal derivative of the potential is specified. We have already obtained the formal solution to such problems by the method of Green's functions. To find such functions, however, can be quite complicated, and several methods have been developed to solve boundary-value problems in electrostatics. In this chapter we will see some of them.

2.1 Method of images

Charge in the presence of a conductor plane

Imagine we have a conductor plane at $z = 0$, which forces the potential to vanish there, *.i.e.* $\Phi(x, y, 0) = 0$, and we place a point charge of magnitude q at a distance d from the plane (see figure below on the left). We want to find the potential in the region $z > 0$.



Let us locate the charge at $(0, 0, d)$ for simplicity. The method of images consist on adding *image* charges *outside* the region of interest in such a way that the boundary conditions are satisfied. The potential is then the sum of the potentials for the point charge (or charges) and their images. In the present case, it is enough to add a single image charge located at $(0, 0, -d)$ and of magnitude $-q$. The total potential is

$$\Phi(x, y, z) = \frac{q}{4\pi\epsilon_0}\frac{1}{(x^2 + y^2 + (z - d)^2)^{1/2}} - \frac{q}{4\pi\epsilon_0}\frac{1}{(x^2 + y^2 + (z + d)^2)^{1/2}} \tag{44}$$

the first term is simply the potential due to the point charge at $(0, 0, d)$ while the second term is the potential of the *image charge*, which we added in order for the boundary conditions at $z = 0$ to be satisfied. Indeed, you can check that the above potential satisfies both the Poisson equation and the correct boundary conditions

$$\nabla^2\Phi(x, y, z) = -\frac{q}{\epsilon_0}\delta(x)\delta(y)\delta(z - d), \qquad \text{for } z > 0, \tag{45}$$

$$\Phi(x, y, 0) = 0 \tag{46}$$

Note that it is important that the image charge is located *outside* the region of interest (see figure above on the right), hence the total potential will still satisfy the Poisson equation in the region of interest. More precisely, we could have added the delta function corresponding to the image charge, but this vanishes identically for $z > 0$.

The method of images is intimately connected to the method of Green's functions. In order to make this relation clear, let us solve this problem by using the method of Green's functions. The Green's functions for Dirichlet boundary conditions on the plane $z = 0$ take the form

$$G(\vec{r}, \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|} + F(\vec{r}, \vec{r}'), \quad \nabla'^2 F(\vec{r}, \vec{r}') = 0, \quad z, z' > 0 \tag{47}$$

where $F(\vec{r}, \vec{r}')$ has to be chosen such as to satisfy

$$G(\vec{r}, \vec{r}') = 0, \qquad \text{for } z' = 0. \tag{48}$$

we can choose

$$F(\vec{r}, \vec{r}') = -\frac{1}{|\vec{r} - \vec{r}'_R|} \tag{49}$$

where $\vec{r}'_R$ is the reflection of $\vec{r}'$ on the plane, namely $\vec{r}'_R = (x', y', -z')$. Note that in the region of interest the Laplacian of $F(\vec{r}, \vec{r}')$ vanishes. The relation with the method of images is now clear, the Green's function for Dirichlet boundary conditions on $z = 0$

$$G_D(\vec{r}, \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|} - \frac{1}{|\vec{r} - \vec{r}'_R|} \tag{50}$$

is simply the potential for a point charge of unit magnitude (in units of $4\pi\epsilon_0$) in the presence of a conductor plane at $z = 0$, where $\vec{r}'$ is the location of the point charge. This has to be the case, indeed, by definition the Green's function satisfies the Poisson equation with a unit charge as a source and vanishes at $z = 0$. [2]

Having constructed the appropriate Green's function we can now plug the desired charge distribution in (40). For example, for a point charge at $\vec{d}$ we obtain

$$\Phi(\vec{r}) = q \int_V \delta(\vec{r}' - \vec{d}) \left(\frac{1}{|\vec{r} - \vec{r}'|} - \frac{1}{|\vec{r} - \vec{r}'_R|} \right) dV' = \frac{q}{|\vec{r} - \vec{d}|} - \frac{q}{|\vec{r} - \vec{d}_R|} \tag{51}$$

exactly as expected from the method of images.

Returning to the potential (44) there are many questions we can ask. For instance, what is the force acting on the charge q? the simplest way to calculate this is by computing the force done on the charge q by its image, an attractive force along the z direction with magnitude

$$|F_z| = \frac{1}{4\pi\epsilon_0} \frac{q^2}{4d^2} \tag{52}$$

[2] Note that the Green's function is symmetric under interchange of $\vec{r}$ and $\vec{r}'$, so it also vanishes for $z' = 0$, which was the original requirement for Green's functions in section one.

There is another, more instructive, way to compute the force. Though the potential vanishes at $z = 0$ its normal derivative does not:

$$\frac{\partial}{\partial z}\Phi|_{z=0} = \frac{1}{2\pi\epsilon_0}\frac{q\,d}{(x^2+y^2+d^2)^{3/2}} \tag{53}$$

Through (31) we can interpret this result as saying that the charge is inducing a surface density:

$$\sigma(x,y) = -\frac{1}{2\pi}\frac{q\,d}{(x^2+y^2+d^2)^{3/2}} \tag{54}$$

An infinitesimal piece of surface $dxdy$ possesses total charge $\sigma(x,y)dxdy$, hence the total force over the charge is

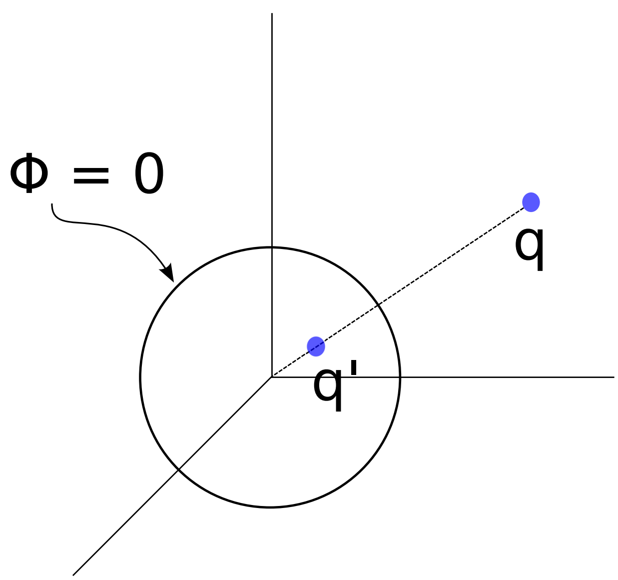
$$F_z = \frac{q}{4\pi\epsilon_0}\int \sigma(x,y)\frac{d}{(x^2+y^2+d^2)^{3/2}}dxdy = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{4d^2} \tag{55}$$

In perfect agreement with our previous result.

In the following we will discuss another geometry where it is possible to guess the location of the image charges.

Charge in the presence of a conductor sphere

Imagine we place a point charge of magnitude q at a distance b from the center of a sphere of radius $a < b$. The sphere is a grounded conductor, so we require the potential to vanish on its surface, namely $\Phi(\vec{r}) = 0$ for $|\vec{r}| = a$. We want to find the potential for this configuration in the region $|\vec{r}| > a$ (see figure).



Let us denote the vector position of the point charge as $\vec{r}_q = \mathbf{n}_q b$ and let us assume that only one image charge of magnitude q' is enough. By symmetry, the image charge should lie on the ray from the origin to the original point charge. The scalar potential would then be

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{|\vec{r}-\mathbf{n}_q b|} + \frac{q'}{|\vec{r}-\mathbf{n}_q b'|}\right) \tag{56}$$

Now we need to adjust q' and b' such that

$$\Phi(\mathbf{n}a) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{|\mathbf{n}a-\mathbf{n}_q b|} + \frac{q'}{|\mathbf{n}a-\mathbf{n}_q b'|}\right) = 0 \tag{57}$$

for any unit vector $\mathbf{n}$. You can explicitly check that this can be achieved by choosing

$$q' = -\frac{a}{b}q, \quad b' = \frac{a^2}{b} \tag{58}$$

Which gives the full potential for the problem at hand

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{|\vec{r}-\mathbf{n}_q b|} - \frac{a}{b}\frac{q}{|\vec{r}-\mathbf{n}_q a^2/b|}\right) \tag{59}$$

Green's function for the sphere

As for the case of the conductor plane, the Dirichlet Green's function for the sphere is simply the potential for a unit charge (including its image) in units of $4\pi\epsilon_0$, where $\vec{r}\,'$ refers to the location of the unit charge. We obtain

$$G(\vec{r},\vec{r}\,') = \frac{1}{|\vec{r}-\vec{r}\,'|} - \frac{a}{r\,'}\frac{1}{|\vec{r}-\frac{a^2}{r'^2}\vec{r}\,'|} = \frac{1}{(r^2+r'^2-2rr'\cos\gamma)^{1/2}} - \frac{1}{\left(\frac{r^2r'^2}{a^2}+a^2-2rr'\cos\gamma\right)^{1/2}} \tag{60}$$

where γ is the angle between $\vec{r}$ and $\vec{r}\,'$. With the Green's function at our disposal, we can write down the general solution to the Dirichlet problem on the sphere. For instance, suppose for simplicity that there is no charge distribution and the prescribed boundary conditions at the sphere fix $\Phi(a,\theta,\phi) = V(\theta,\phi)$, where we are using spherical coordinates. Then, from (40), the solution to the potential takes the form

$$\Phi(\vec{r}) = \frac{1}{4\pi}\int V(\theta',\phi')\frac{a(r^2-a^2)}{(r^2+a^2-2ar\cos\gamma)^{3/2}}\sin\theta' d\theta' d\phi' \tag{61}$$

2.2 Method of orthonormal functions

A very important feature of the equations of electrostatics, (17) and (23) or equivalently (24), is that they are linear equations (in the fields or the potential). This leads to the principle of superposition: the sum of solutions is also a solution. Conversely any solution can also be expressed in a convenient basis of solutions (you have already met this phenomenon, when solving the equations of motion for a vibrating string).

The representation of solutions of potential problems by expansions in orthonormal functions is a powerful technique that can be used in a large class of problems. The particular set chosen depends on the symmetries of the problem.

Generalities

We start by recalling some general properties of orthonormal functions. Consider an interval (a, b) and a set of real or complex functions in one variable $U_n(x)$, $n = 1, 2, ...$ The orthonormality condition takes the form

$$\int_a^b U_m^*(x) U_n(x) dx = \delta_{mn} \tag{62}$$

If the above set is complete then an arbitrary function $f(x)$ (square integrable in the interval (a, b)) can be written as a linear combination of the orthonormal functions

$$f(x) = \sum_{n=1}^{\infty} a_n U_n(x) \tag{63}$$

where the coefficients are given by

$$a_n = \int_a^b U_n^*(x) f(x) dx \tag{64}$$

Plugging this expression into (63) we obtain

$$f(x) = \int_a^b \left(\sum_{n=1}^{\infty} U_n^*(x') U_n(x) \right) f(x') dx' \tag{65}$$

recalling the properties of the Dirac delta function, it should be the case that

$$\sum_{n=1}^{\infty} U_n^*(x') U_n(x) = \delta(x' - x) \tag{66}$$

This is called completeness or closure relation. The most famous example of orthogonal functions are the sines and cosines that enter in the Fourier expansion. For the interval $(-a/2, a/2)$ they are

$$\sqrt{\frac{2}{a}} \sin\left(\frac{2\pi n x}{a}\right), \quad \sqrt{\frac{2}{a}} \cos\left(\frac{2\pi n x}{a}\right) \tag{67}$$

The extension to higher dimensions is obvious. For instance, a function of two variables admits an expansion in terms of the sets $U_n(x)$ and $V_n(y)$.

$$f(x,y) = \sum a_{mn} U_m(x) V_n(y), \quad \text{where} \;\; a_{mn} = \int dx \int dy \, U_m^*(x) V_n^*(y) f(x,y) \tag{68}$$

If the interval becomes infinite the set of orthogonal functions $U_n(x)$ may become an uncountable set, namely $U_n(x) \to U_k(x)$ with k a real number. In this case sums are replaced by integrals and the Kronecker delta is replaced by the Dirac delta. The most famous example is the exponential Fourier series expansion in terms of the set

$$U_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx} \tag{69}$$

An arbitrary function can be expanded in terms of those

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) e^{ikx} dk \tag{70}$$

This expansion is usually called exponential Fourier expansion. The "coefficients" $A(k)$ (which now depend on a continuous function) are given by

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx \tag{71}$$

The orthonormality condition is

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(k-k')x} dx = \delta(k-k') \tag{72}$$

while the completeness relation takes the form

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} dk = \delta(x-x') \tag{73}$$

These last two integrals give useful representations of the delta function, and will be used below.

Cartesian coordinates

For electrostatic problems with rectangular symmetries it is convenient to use cartesian coordinates. The Laplace equation in cartesian coordinates is

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \tag{74}$$

Looking for separable solutions of the form $\Phi(x,y,z) = X(x)Y(y)Z(z)$ we find

$$\frac{1}{X(x)} \frac{d^2 X}{dx^2} + \frac{1}{Y(y)} \frac{d^2 Y}{dy^2} + \frac{1}{Z(x)} \frac{d^2 Z}{dz^2} = 0 \tag{75}$$

which implies

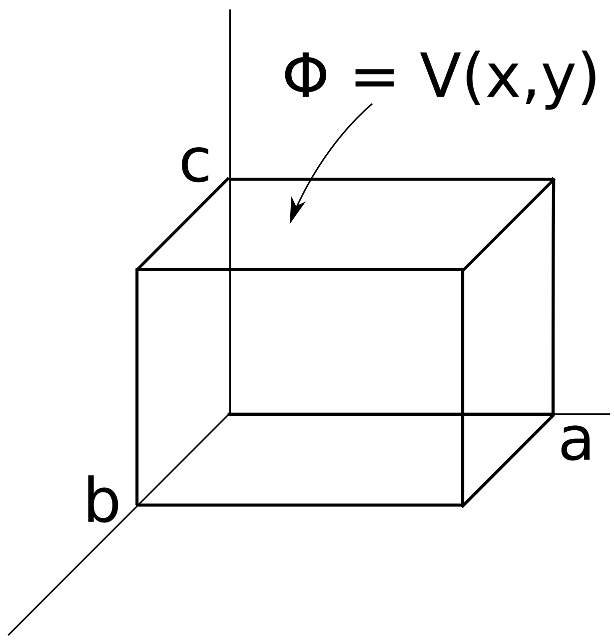
$$\frac{1}{X(x)}\frac{d^2X}{dx^2} = -\alpha^2, \qquad \frac{1}{Y(y)}\frac{d^2Y}{dy^2} = -\beta^2, \qquad \frac{1}{Z(x)}\frac{d^2Z}{dz^2} = \alpha^2 + \beta^2 \tag{76}$$

Hence we find the following basis of solutions

$$\Phi = e^{\pm i\alpha x} e^{\pm i\beta y} e^{\pm\sqrt{\alpha^2+\beta^2}z} \tag{77}$$

where at this point α and β are arbitrary, but we have chosen them to be real. Any linear combination of the above solutions is a solution. When specific boundary conditions are imposed on the potential, we will have a restriction on the values that α and β can take.

For example, let us consider the problem of finding the potential inside a rectangular box with dimensions (a, b, c) in the (x, y, z) directions, such that the potential vanishes at all faces except at $z = c$, where it takes some prescribed value $\Phi(x, y, c) = V(x, y)$, see figure.



Requiring the potential to vanish at $x = 0, y = 0$ and $z = 0$ we see the potential should be the sum of terms of the form

$$\Phi = \sin\alpha x \sin\beta y \sinh\sqrt{\alpha^2+\beta^2}z \tag{78}$$

Requiring the potential to vanish at $x = a$ and $y = b$ further fixes the form of α and β

$$\alpha = \frac{m\pi}{a}, \quad \beta = \frac{n\pi}{b} \tag{79}$$

for m, n integer numbers. Hence, requiring the boundary conditions on five of the six faces fixes the form of the solution to be

$$\Phi(x,y,z) = \sum_{m,n=1}^{\infty} A_{m,n} \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) \sinh\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} z \tag{80}$$

The remaining boundary condition requires

$$\sum_{m,n=1}^{\infty} A_{m,n} \sin\left(\frac{m\pi}{a}x\right) \sin\frac{n\pi}{b}y \sinh\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} c = V(x,y) \tag{81}$$

Which is basically the sine Fourier expansion of $V(x,y)$. Hence the coefficients $A_{m,n}$ are given by

$$A_{m,n} = \frac{4}{ab \sinh\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} c} \int_0^a dx \int_0^b dy \sin\frac{m\pi}{a}x \sin\frac{n\pi}{b}y V(x,y) \tag{82}$$

where we have considered an odd extension of V[3] and used the orthonormality of the functions entering the solution:

$$\int_{-a}^{a} dx \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{m'\pi}{a}x\right) = a\delta_{mm'} \tag{83}$$

for $m \neq 0$.

Another problem which can be easily solved by the method of rectangular orthogonal functions is that of a point charge q inside a grounded box. Imagine the potential is required to vanish at the six faces $x, y, z = 0$ and $x = a, y = b, z = c$, and we locate a charge q at (x_0, y_0, z_0). The potential should satisfy the corresponding Poisson equation

$$\nabla^2\Phi(x,y,z) = -\frac{q}{\epsilon_0}\delta(x - x_0)\delta(y - y_0)\delta(z - z_0) \tag{84}$$

The boundary conditions suggest we expand the solution in terms of the set of orthogonal functions

$$\Phi(x,y,z) = \sum_{m,n,\ell=1}^{\infty} A_{m,n,\ell}\sqrt{\frac{8}{abc}} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sin\left(\frac{\ell\pi z}{c}\right) \tag{85}$$

Note that the boundary conditions are automatically satisfied. Plugging this expression into the right hand side of the Poisson equation (84) we find

$$-\sum_{m,n,\ell=1}^{\infty} A_{m,n,\ell}\sqrt{\frac{8}{abc}} \left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{\ell\pi}{c}\right)^2\right) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sin\left(\frac{\ell\pi z}{c}\right) \tag{86}$$

[3]Remember from Fourier series, in order to do a sine expansion often you have to extend your functions to the region $(-a, 0)$ in a fashion consistent with the odd symmetry of $\sin x$.

On the other hand, using the completeness relation (66) for the orthonormal functions at hand gives the following representation of the delta function

$$\delta(x-x_0) = \sum_{m=1}^{\infty} \frac{2}{a} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{m\pi x_0}{a}\right) \tag{87}$$

and similar for $\delta(y-y_0)$ and $\delta(z-z_0)$. We see the Poisson equation is satisfied if

$$A_{m,n,\ell} = \frac{q}{\epsilon_0}\sqrt{\frac{8}{abc}}\left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{\ell\pi}{c}\right)^2\right)^{-1} \sin\left(\frac{m\pi x_0}{a}\right) \sin\left(\frac{n\pi y_0}{b}\right) \sin\left(\frac{\ell\pi z_0}{c}\right)$$

which gives the final solution

$$\Phi(x,y,z) = \frac{q}{\epsilon_0}\sum_{m,n,\ell=1}^{\infty} \frac{8}{abc} \frac{\sin\left(\frac{m\pi x_0}{a}\right)\sin\left(\frac{n\pi y_0}{b}\right)\sin\left(\frac{\ell\pi z_0}{c}\right)}{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{\ell\pi}{c}\right)^2} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sin\left(\frac{\ell\pi z}{c}\right)$$

It would have been extremely hard to solve this problem by other methods!

Finally, in order to solve problems with boundary surfaces it is sometimes convenient to expand the corresponding Green's function in the particular set of orthonormal functions under consideration. For instance, lets consider Dirichlet boundary conditions at the plane $z=0$. We have already found the Green's function for this problem and have obtained

$$G(\vec{r},\vec{r}\,') = \frac{1}{|\vec{r}-\vec{r}\,'|} - \frac{1}{|\vec{r}-\vec{r}\,'_R|} \tag{88}$$

In order to expand this in rectangular orthonormal functions, recall the integral representation (73) for the Dirac delta function

$$\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} d\gamma e^{i\alpha x} e^{i\beta y} e^{i\gamma z} = \delta(x)\delta(y)\delta(z) \tag{89}$$

This suggests

$$\frac{-4\pi}{(2\pi)^3}\int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} d\gamma \frac{e^{i\alpha x} e^{i\beta y} e^{i\gamma z}}{\alpha^2+\beta^2+\gamma^2} = \frac{1}{(x^2+y^2+z^2)^{1/2}} \tag{90}$$

Since acting with the Laplacian would produce $-4\pi\delta(\vec{r})$ on both sides. Integrating over γ we obtain

$$\frac{1}{(x^2+y^2+z^2)^{1/2}} = -\frac{1}{2\pi}\int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\beta e^{i\alpha x} e^{i\beta y} \frac{e^{-|z|\sqrt{\alpha^2+\beta^2}}}{\sqrt{\alpha^2+\beta^2}} \tag{91}$$

Hence we obtain for the Green's function

$$\frac{1}{|\vec{r}-\vec{r}\,'|} - \frac{1}{|\vec{r}-\vec{r}\,'_R|} = -\frac{1}{2\pi}\int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\beta \frac{e^{i\alpha(x-x')} e^{i\beta(y-y')}}{\sqrt{\alpha^2+\beta^2}} \left(e^{-|z-z'|\sqrt{\alpha^2+\beta^2}} - e^{-|z+z'|\sqrt{\alpha^2+\beta^2}}\right) \tag{92}$$

As expected, note that the r.h.s. vanishes if either $z = 0$ or $z' = 0$.

Spherical coordinates

The Laplace equation in spherical coordinates (r, θ, ϕ) takes the form

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0 \quad (93)$$

Note that the Laplacian splits into a "radial" part and an "angular" part

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2}\nabla^2_{\theta,\phi}\Phi \quad (94)$$

$$\nabla^2_{\theta,\phi} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} \quad (95)$$

It is convenient to look for separable solutions of the form

$$\Phi(r, \theta, \phi) = R(r)Y(\theta, \phi) \quad (96)$$

Proceeding as before, we find the expansion in terms of orthonormal functions takes the form

$$\Phi = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(A_{\ell,m}r^{\ell} + B_{\ell,m}r^{-(\ell+1)}\right)Y_{\ell m}(\theta, \phi) \quad (97)$$

The orthonormal functions $Y_{\ell m}(\theta, \phi)$ depend only on the angular variables and are called spherical harmonics. Here $\ell = 0, 1, 2, ...$ and for a fixed ℓ , m takes the integer values from $-m$ to m. They satisfy

$$\nabla^2_{\theta,\phi}Y_{\ell m}(\theta, \phi) = -\ell(\ell+1)Y_{\ell m}(\theta, \phi) \quad (98)$$

This is a complete set of functions, in the sense that any (single valued!) function of the angular variables $g(\theta, \phi)$ can be written in terms of those. They satisfy

$$Y_{\ell,-m}(\theta, \phi) = (-1)^m Y^*_{\ell,m}(\theta, \phi) \quad (99)$$

The orthonormality condition takes the form

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta Y^*_{\ell',m'}(\theta, \phi)Y_{\ell,m}(\theta, \phi) = \delta_{\ell,\ell'}\delta_{m,m'} \quad (100)$$

while the completeness relation takes the form

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y^*_{\ell,m}(\theta', \phi')Y_{\ell,m}(\theta, \phi) = \delta(\phi - \phi')\delta(\cos\theta - \cos\theta') \quad (101)$$

The first few spherical harmonics are

$$Y_{00} = \frac{1}{\sqrt{4\pi}} \tag{102}$$

$$Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi} \tag{103}$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta \tag{104}$$

$$Y_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\theta e^{2i\phi} \tag{105}$$

$$Y_{21} = = -\sqrt{\frac{15}{8\pi}} \sin\theta\cos\theta e^{i\phi} \tag{106}$$

$$Y_{20} = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) \tag{107}$$

The prototypical problem with spherical symmetry is that of finding the potential *inside* a sphere of radius a with prescribed potential at the sphere. Namely the potential satisfies the Laplace equation and the boundary conditions

$$\Phi(a,\theta,\phi) = V(\theta,\phi) \tag{108}$$

$$\Phi(r,\theta,\phi) \quad \text{is bounded as } r \to 0 \tag{109}$$

Where $V(\theta,\phi)$ is some prescribed function. The general solution to this problem takes the form

$$\Phi(r,\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_{\ell,m} r^{\ell} Y_{\ell m}(\theta,\phi) \tag{110}$$

where we have used the fact that the solution is bounded as $r \to 0$, so as to keep only non-negative powers or r. Note that if we were interested in solving the problem *outside* the sphere, we would require that the potential is bounded at infinity, and we would keep only the negative powers. The coefficients $A_{\ell,m}$ can be fixed by requiring the correct boundary conditions at $r = a$:

$$\Phi(a,\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_{\ell,m} a^{\ell} Y_{\ell m}(\theta,\phi) = V(\theta,\phi) \tag{111}$$

Using the orthonormality condition (100) we obtain

$$A_{\ell,m} = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta Y^*_{\ell,m}(\theta,\phi) V(\theta,\phi) \tag{112}$$

As in the case of rectangular coordinates, for problems including boundaries, it will be useful to expand the Dirichlet Green's function for a sphere of radius a in terms of spherical orthonormal functions

$$G(\vec{r},\vec{r}\,') = \sum_{\ell,m} A_{\ell,m} Y_{\ell,m}(\theta,\phi) \tag{113}$$

where (θ,ϕ) are the angular coordinates of the point r and the "coefficients" $A_{\ell,m}$ depend on (r,r',θ',ϕ'). Since the l.h.s is symmetric under interchange of $\vec{r}$ and $\vec{r}\,'$ we can furthermore write

$$G(\vec{r},\vec{r}\,') = \sum_{\ell,m} Y^*_{\ell,m}(\theta',\phi') A_{\ell,m}(r,r') Y_{\ell,m}(\theta,\phi) \tag{114}$$

In order to compute the coefficients $A_{\ell,m}(r,r')$ we can proceed as follows. The Dirac delta function in spherical coordinates takes the form [4]

$$\delta(\vec{r}-\vec{r}\,') = \frac{1}{r^2}\delta(r-r')\delta(\phi-\phi')\delta(\cos\theta-\cos\theta') \tag{115}$$

Using the completeness relation (101) we can write

$$\delta(\vec{r}-\vec{r}\,') = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{1}{r^2}\delta(r-r') Y^*_{\ell,m}(\theta',\phi') Y_{\ell,m}(\theta,\phi) \tag{116}$$

hence

$$\nabla^2 \sum_{\ell,m} Y^*_{\ell,m}(\theta',\phi') A_{\ell,m}(r,r') Y_{\ell,m}(\theta,\phi) = -4\pi \sum_{\ell,m}^{\infty} \frac{1}{r^2}\delta(r-r') Y^*_{\ell,m}(\theta',\phi') Y_{\ell,m}(\theta,\phi) \tag{117}$$

which implies

$$\frac{1}{r}\frac{d^2}{dr^2}(rA_{\ell,m}) - \frac{\ell(\ell+1)}{r^2}A_{\ell,m} = -\frac{4\pi}{r^2}\delta(r-r') \tag{118}$$

For the regions $r > r'$ and $r < r'$ the delta function vanishes and we obtain

$$A_{\ell,m}(r,r') = \begin{cases} Ar^\ell + Br^{-(\ell+1)}, & r < r' \\ \\ A'r^\ell + B'r^{-(\ell+1)}, & r > r' \end{cases} \tag{119}$$

Now we need to fix the constants of integration A, B, A', B' (careful: since the equation is on r, these constants could actually depend on r'). Remember that we interpret $\vec{r}$ as the observation point and $\vec{r}\,'$ as the location of the unit point charge. Since the Green's function

[4] To express $\delta(\mathbf{x}-\mathbf{x}') = \delta(x_1-x_1')\delta(x_2-x_2')\delta(x_3-x_3')$ in terms of coordinates another set of coordinates (y_1,y_2,y_3) we need to divide by the Jacobian, so that $\delta(\mathbf{x}-\mathbf{x}')d^3x$ is invariant.

is bounded as we take the observation point to infinity, we need $A' = 0$. Furthermore, the Green's function vanishes as $r = a$, which gives a relation between A and B and continuity at $r = r'$ leave us only with an overall constant

$$A_{\ell,m}(r, r') = \begin{cases} A\left(r^\ell - \frac{a^{2\ell+1}}{r^{(\ell+1)}}\right), & r < r' \\ \\ A\left(r'^{2\ell+1} - a^{2\ell+1}\right) r^{-(\ell+1)}, & r > r' \end{cases} \tag{120}$$

The overall constant can be fixed as follows. Multiplying (118) by r and integrating both sides over the interval $r = r' - \epsilon$ to $r = r' + \epsilon$ we obtain

$$\left[\frac{d}{dr}\left(rA_{\ell,m}(r, r')\right)\right]_{r'+\epsilon} - \left[\frac{d}{dr}\left(rA_{\ell,m}(r, r')\right)\right]_{r'-\epsilon} = -\frac{4\pi}{r'} \tag{121}$$

from where it follows

$$A = 4\pi \frac{r'^{-(\ell+1)}}{2\ell + 1} \tag{122}$$

Putting all together we can write the Green's function as

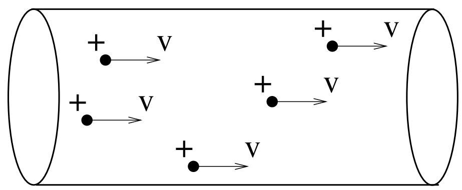
$$G(\vec{r}, \vec{r}') = 4\pi \sum_{\ell,m} \frac{1}{2\ell + 1} \left(r_<^\ell - \frac{a^{2\ell+1}}{r_<^{\ell+1}}\right) r_>^{-(\ell+1)} Y^*_{\ell,m}(\theta', \phi') Y_{\ell,m}(\theta, \phi) \tag{123}$$

where $r_>$ and $r_<$ denote the bigger and smaller between r and r'. Note that the Green's function vanishes for both $r = a$ and $r' = a$ (when one of the the two conditions is satisfied, the other radius is bigger, since we are looking at the sphere from outside). In very much the same way one can work out The Green's function for the *interior* of the sphere. In this case we obtain

$$G(\vec{r}, \vec{r}') = 4\pi \sum_{\ell,m} \frac{1}{2\ell + 1} r_<^\ell \left(r_>^{-(\ell+1)} - \frac{r_>^\ell}{a^{2\ell+1}}\right) Y^*_{\ell,m}(\theta', \phi') Y_{\ell,m}(\theta, \phi) \tag{124}$$

3 Magnetostatics

Frequently, people find magnetism a little more mysterious than electricity. Rather than thinking of fridge magnets, you should think of electromagnets: charges in motion give rise to electric currents and these produce magnetic fields; many of you will have visualized these fields in experiments by sprinkling iron filings on sheets of cardboard transverse to a current-carrying wire. (The fridge magnet, or any permanent magnet, derives its magnetism from microscopic currents.) The subject corresponding to electrostatics, which is produced by time-independent charges, is magnetostatics, which is produced by time-independent, or "steady", currents.



Given a collection of point charges Q_i in motion with velocities $\vec{v}_i$, we define the corresponding electric currents as:

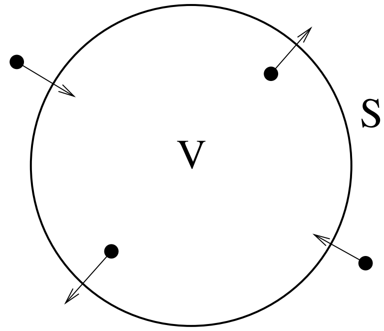
$$\vec{j} = \sum q_i \vec{v}_i$$

This is a vector field made by adding up elementary contributions. By analogy with charge density ρ as a smoothed-out distribution of point charges, we introduce a vector field, the current density $\vec{J}(x, y, z, t)$, so that the total electric current in a region V is the integral $\int_V \vec{J} dV$.

In this section we shall usually be thinking of steady currents, but there is one thing to deal with first, namely the mathematical expression for the physical observation than charge is conserved.

3.1 Conservation of charge

Consider a region V with surface S with charges moving through.



The total charge inside V is $Q = \int_V \rho dV$, so that

$$\begin{aligned}
\frac{dQ}{dt} &= \int_V \frac{\partial \rho}{\partial t} dV \qquad \text{allowing } \tfrac{\partial \rho}{\partial t} \neq 0 \text{ for the moment} \\
&= \text{rate of increase of } Q \\
&= \text{rate charge goes in} - \text{rate charge goes out} \\
&= -\int_S \vec{J} \cdot d\mathbf{S} \\
&= -\int_V \nabla \cdot \vec{J} dV \qquad \text{by divergence theorem}
\end{aligned}$$

an so

$$\int_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} \right) dV = 0 \tag{125}$$

this is to be true for all regions V, then

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0 \tag{126}$$

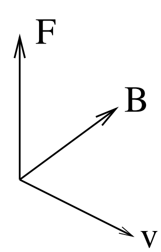
which is the *charge conservation equation.*

For the rest of this section, we suppose none of the quantities of interest depends on time. For this time independent situation (126) reduces to $\nabla \cdot \vec{J} = 0$.

3.2 Magnetic Field Strength

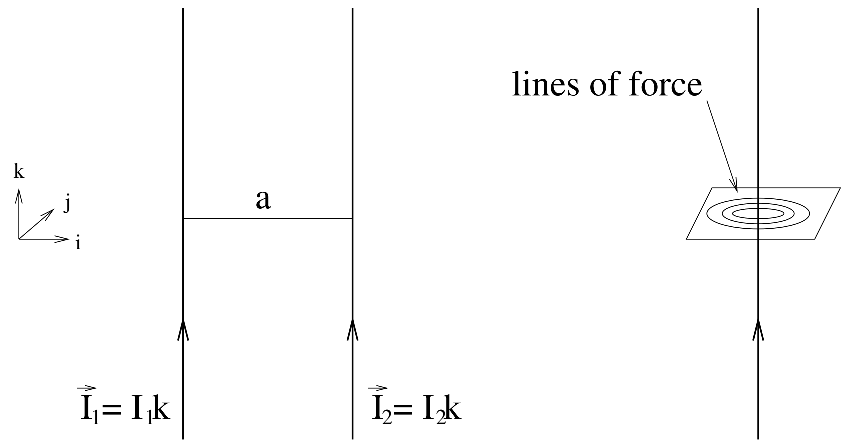
Like the electric field $\vec{E}$, the magnetic field strength $\vec{B}$ can be defined from an experimentally stablished force law. Analogously to (7), a charged particle with charge q, moving with velocity $\vec{v}$ in a magnetic field $\vec{B}$ is subject to a force

$$\vec{F} = q\vec{v} \wedge \vec{B} \tag{127}$$



This serves to define $\vec{B}$, but note a peculiar feature of this force law, that the force is orthogonal to the velocity. In particular, therefore $\vec{v} \cdot \vec{F} = 0$ and the force does no work.

Force between two parallel current-carrying wires



To see (127) in action, let us consider the force between two parallel current-carrying wires.

What can we say about the magnetic field $\vec{B}$ generated by a single wire? let us choose coordinates such that the wire is along the z direction and let us use cylindrical-polar coordinates. It can be experimentally verified that if we take a charge q and move it around the wire, the charge is subject to a force. This force vanishes, *i.e.* there is no force, if the charge moves along the θ direction, while there is a force if the charge moves in any other direction. This means that $\vec{B}$ has a component only in the θ direction[5]; using the symmetry in θ and z we can write $\vec{B} = B(R)\vec{e}_\theta$, where R is the distance to the wire.

Now suppose we have parallel wires, wire 1 carrying current I_1 and wire 2 carrying current I_2. Each wire gives rise to a B-field and the field from one exerts a force on the current in the other. Using (127), and thinking of the currents as charged particles in motion, we have:

Force per unit length on wire 2 due to wire 1 is

$$\vec{F}_{12} = \vec{I}_2 \wedge \vec{B}_1.$$

where $\vec{B}_1$ means $\vec{B}$ at wire 2 due to current in wire 1, and from the geometry $\vec{I}_2 = I_2\vec{k}$. Similarly, the force on wire 1 due to wire 2 is

$$\vec{F}_{21} = \vec{I}_1 \wedge \vec{B}_2.$$

where $\vec{B}_2$ is $\vec{B}$ at wire 1 due to wire 2, and $\mathbf{I}_1 = I_1\vec{k}$. These forces must be equal in magnitude and opposite, due to Newton's third law:

[5]This is consistent with the following fact: if you sprinkle iron filings on a cardboard held at right angle to a current-carrying wire, you can see that the magnetic lines of force are concentric circles, centered on the wire.

$$\vec{I}_2 \wedge \vec{B}_1 = -\vec{I}_1 \wedge \vec{B}_2.$$

From what was said about the direction of $\vec{B}$, we have that at wire 2, $\vec{B}_1 = \vec{j} B_1$ in terms of some magnitude B_1, and in wire 1, $\vec{B}_2 = -\vec{j} B_2$ in terms of some magnitude B_2. Therefore

$$|\vec{F}_{12}| = I_2 B_1 = |F_{21}| = I_1 B_2.$$

It is an experimental fact that this magnitude is

$$|\vec{F}| = \frac{\mu_0}{2\pi a} I_1 I_2$$

in terms of a constant μ_0 (which, as with ϵ_0 is needed to get the right dimensions).

From this experimental fact we deduce that, at wire 2,

$$\vec{B} = \frac{\mu_0 I_1}{2\pi a} \vec{j}$$

At a general point, in terms of cylindrical polar coordinates (R, θ, z) with the wire along the z-axis this is

$$= \frac{\mu_0 I}{2\pi} \frac{1}{R} \vec{e}_\theta$$

which we can write in cartesian coordinates as

$$\vec{B} = \frac{\mu_0 I}{2\pi} \left(-\frac{y}{R^2}, \frac{x}{R^2}, 0 \right), \qquad R^2 = x^2 + y^2 \tag{128}$$

This is the magnetic field due to an infinite straight wire with constant current, which can be though of as the elementary magnetic field, much as (4), $\vec{E} = \frac{Q}{4\pi\epsilon_0} \frac{\vec{r}}{r^3}$ gives the elementary electric field.

3.3 Differential equations for the magnetic field

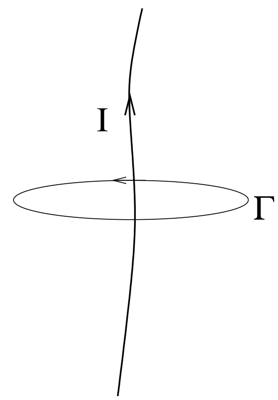
Our aim for the next few pages is to calculate the $\vec{B}$ due to current in an arbitrary wire, much as the integral (26) together with the relation $\vec{E} = -\nabla\Phi$ gives $\vec{E}$ for an arbitrary charge distribution. We achieve this aim with (137) below. As a step towards this aim, we note that, when Γ is a horizontal circle of radius R centered on a straight wire

$$\oint_\Gamma \vec{B} \cdot d\vec{\ell} = \frac{\mu_0 I}{2\pi} \oint \frac{1}{R} \vec{e}_\theta \cdot \vec{e}_\theta R d\theta \tag{129}$$

$$= \mu_0 I \tag{130}$$

Even though we have derived this for a straight wire, the result is much more general:

Ampere's Law: $\oint_\Gamma \vec{B} \cdot d\vec{\ell} = \mu_0 \times$ total current through Γ.



Compare ampere's law to Gauss Law in the first form we had.

For a current density instead of a wire, we obtain an integral version of Ampere's Law:

$$\begin{aligned} \oint_\Gamma \vec{B} \cdot d\vec{\ell} &= \mu_0 \int_\Sigma \vec{J} \cdot d\mathbf{S} \qquad \Sigma \text{ spans } \Gamma \\ &= \int_\Sigma \nabla \wedge \vec{B} \cdot d\mathbf{S} \qquad \text{by Stoke's theorem} \end{aligned}$$

and for this to be true for all Σ we obtain the differential version of Ampere's law:

$$\nabla \wedge \vec{B} = \mu_0 \vec{J} \tag{131}$$

(recall that $\nabla \cdot \vec{J} = 0$ from charge conservation and time independence, and we need this for (131) to make sense). This equation is supplemented by

$$\nabla \cdot \vec{B} = 0 \tag{132}$$

Which can be explicitly checked for the straight wire, from (128). Equation (132) is to be contrasted with (23): $\nabla \cdot \vec{E} = \frac{1}{\epsilon_0}\rho$. Thus we interpret (132) as saying *"there are no magnetic charges"*.

The magnetic potential

It follows from (17) that, in a simply-connected region, there exists a function Φ such that $\vec{E} = -\nabla\Phi$. It is a less familiar fact that

Claim: *if $\nabla \cdot \vec{B} = 0$ in a suitable region, then there exists in that region a vector field $\vec{A}$ such that*

$$\vec{B} = \nabla \wedge \vec{A}$$

.

$\vec{A}$ is the *magnetic potential* (also called *vector* potential, in which case Φ is called the *scalar* potential).

Note that a change $\vec{A} \to \vec{A} + \nabla\zeta$, for any scalar function ζ, leaves $\vec{B}$ unchanged. We may exploit this freedom to impose another condition, namely

$$\nabla \cdot \vec{A} = 0$$

for suppose $F = \nabla \cdot \vec{A} \neq 0$ and change $\vec{A} \to \vec{A} + \nabla \cdot \zeta$, then this would change $F \to F + \nabla^2\zeta$; now, choose ζ such that $\nabla^2\zeta = -F$, so that now $\nabla \cdot \vec{A} = 0$.

Equation (131) can be turned into an equation for $\vec{A}$ as follows:

$$\begin{aligned} \nabla \wedge \vec{B} &= \nabla \wedge (\nabla \wedge \vec{A}) = \nabla(\nabla \cdot A) - \nabla^2 \vec{A} \\ &= \mu_0 \vec{J} \end{aligned} \tag{133}$$

with the choice $\nabla \cdot \vec{A} = 0$, then we have

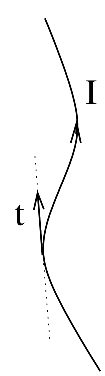
$$\nabla^2 \vec{A} = -\mu_0 \vec{J} \tag{134}$$

Remember that this is true for the particular choice $\nabla \cdot \vec{A} = 0$. This has the vector form of the *Poisson's equation*, and we've solved Poisson's equation before: recall (26) and compare with (24). So we solve (134) by

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} dV'$$

This is for a volume distribution of current. If we have just a line current $\vec{J} = I\vec{t}$, where I is constant and $\vec{t}$ is the tangent (of unit length) to a curve L (where the curve L is the "wire"), then instead

$$\vec{A} = \frac{\mu_0 I}{4\pi} \int_L \frac{\vec{t} d\ell}{|\vec{r} - \vec{r}\,'(\ell)|}$$



where ℓ parametrizes where we are along the wire.

From the vector potential $\vec{A}$ we can calculate $\vec{B}$. For a volume distribution

$$\vec{B}(\vec{r}) = \nabla \wedge \vec{A} = \frac{\mu_0}{4\pi} \nabla \wedge \int \frac{\vec{J}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} dV' \tag{135}$$

$$= \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,') \wedge (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} dV' \tag{136}$$

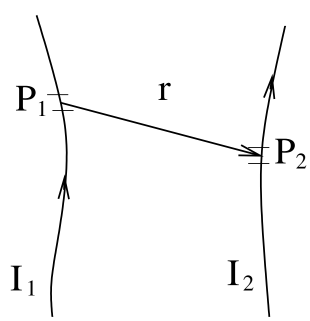
while for a line distribution/wire

$$\vec{B}(\vec{r}) = \nabla \wedge \vec{A} = \frac{\mu_0 I}{4\pi} \int_L \frac{\vec{t} \wedge (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} d\ell$$

or

$$B(\vec{r}) = \frac{\mu_0 I}{4\pi} \int_L \frac{d\vec{\ell} \wedge (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} \tag{137}$$

Where $d\vec{\ell} = d\ell\vec{t}$. This is now $\vec{B}$ at all points of space due to an arbitrary wire, and so generalizes (128). We can use it to obtain an expression for the force between two arbitrary wires, by chopping the second wire into elementary pieces.



At P_2 on wire 2

$$d\vec{F} = I_2 d\vec{\ell}_2 \wedge \vec{B}_1$$

so

$$\vec{F} = \int I_2 d\vec{\ell}_2 \wedge \vec{B}_1$$

$$= \frac{\mu_0 I_1 I_2}{4\pi} \iint \frac{d\vec{\ell}_2 \wedge (d\vec{\ell}_1 \wedge (\vec{r}_2 - \vec{r}_1))}{|\vec{r}_2 - \vec{r}_1|^3} \tag{138}$$

where $\vec{r}_1 = \vec{r}_1(\ell_1)$ belongs to wire 1 and $\vec{r}_2 = r_2(\ell_2)$ belongs to wire 2. This is the *Biot-Savart Law* for the force between two arbitrary current-carrying wires (more precisely, the force on wire 2 done by wire 1). Its interest is largely theoretical, as it is rather hard to use.

3.4 The story so far

We've studied time-independent electricity (or electrostatics) and time-independent magnetism (or magnetostatics), and found these to be governed by the following system of equations

$$\begin{aligned} \nabla \cdot \vec{E} &= \frac{1}{\epsilon_0}\rho \qquad \nabla \cdot \vec{B} = 0 \\ \nabla \wedge \vec{E} &= 0 \qquad \nabla \wedge \vec{B} = \mu_0 \vec{J} \end{aligned} \tag{139}$$

with the understanding that

$$\frac{\partial \vec{E}}{\partial t} = 0 = \frac{\partial \vec{B}}{\partial t} = \frac{\partial \vec{J}}{\partial t} = \frac{\partial \rho}{\partial t}.$$

The electric and magnetic field strengths themselves can be defined from the force laws (7) and (127), which we combine as follows: a particle with charge Q moving with velocity $\vec{v}$ in a combination of an electric field $\vec{E}$ and a magnetic field $\vec{B}$ is subject to the force given by

$$\vec{F} = Q(\vec{E} + \vec{v} \wedge \vec{B}).$$

In addition we have one equation which we expect to be valid also in the time-dependent case, namely the charge conservation equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0. \tag{140}$$

The problem now is how to extend all these equations to the time-dependent case, i.e. how to put back the terms that are zero in the time-independent case. This will be the subject of the next part of this course.

4 Time dependent electromagnetism

4.1 Maxwell equations

Experimental evidence suggests two qualitative facts:

- A time-varying $\vec{B}$ produces an $\vec{E}$.
- A time-varying $\vec{E}$ produces a $\vec{B}$.

This experimental evidence can be summarized quantitatively in the following two integral laws:

$$\oint_\Gamma \vec{E}\cdot d\vec{\ell} = -\frac{d}{dt}\int_\Sigma \vec{B}\cdot d\mathbf{S} \qquad \text{Faraday's law of induction} \tag{141}$$

$$\oint_\Gamma \vec{B}\cdot d\vec{\ell} = \int_\Sigma \mu_0\vec{J}\cdot d\vec{S} + \int_\Sigma \epsilon_0\mu_0\frac{\partial \vec{E}}{\partial t}\cdot d\vec{S} \tag{142}$$

Where Σ spans Γ. Apply Stoke's theorem to both to obtain

$$\int_\Sigma \left(\nabla\wedge\vec{E} + \frac{\partial\vec{B}}{\partial t}\right)\cdot d\vec{S} = 0 \tag{143}$$

$$\int_\Sigma \left(\nabla\wedge\vec{B} - \mu_0\vec{J} - \epsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}\right)\cdot d\vec{S} = 0 \tag{144}$$

and if these hold for all Σ we obtain the

Maxwell equations:

$$\begin{aligned} \nabla\wedge\vec{E} + \frac{\partial\vec{B}}{\partial t} &= 0; \qquad \nabla\wedge\vec{B} &= \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} \\ \nabla\cdot\vec{B} &= 0, \qquad \nabla\cdot\vec{E} &= \frac{1}{\epsilon_0}\rho \end{aligned} \tag{145}$$

with $c^2 = \frac{1}{\epsilon_0\mu_0}$.

These are the *Maxwell's equations*, the basic equations of the new theory of electromagnetism, in which electricity and magnetism are merged.

Note that

- Two of the equations always have zero on the right; the other two contain the sources ρ and $\vec{J}$.

- If all time-derivatives are set to zero, we recover the equations of the previous section.
- The system obtained by setting ρ and $\vec{J}$ to zero is called *source-free Maxwell equations*.
- Regarding ρ and $\vec{J}$ as given, there are eight equations for six unknowns, so there should be a consistency condition.

To see what the consistency condition is, calculate:

$$\begin{aligned}
\frac{\partial \rho}{\partial t} &= \epsilon_0 \frac{\partial}{\partial t}(\nabla \cdot \vec{E}) = \epsilon_0 \nabla \cdot \left(\frac{\partial \vec{E}}{\partial t}\right) \\
&= \frac{1}{\mu_0} \nabla \cdot \left(\nabla \wedge \vec{B} - \mu_0 \vec{J}\right) \\
&= -\nabla \cdot \vec{J}
\end{aligned}$$

which we recognize as the charge conservation equation. From this point of view the charge conservation equation is a consistency condition for the Maxwell's equations. As an exercise, show that there is a second consistency condition but that it is automatically satisfied.

Energy of the Electromagnetic Field

There is a deductive way to approach this question, but we shall rely on intuition and guesswork. We want an energy density (energy-per-unit-volume) which is something like "$1/2 m\vec{v} \cdot \vec{v}$".

After playing around with Maxwell equations for a while, we are lead to consider

$$\mathcal{E} = \frac{1}{2}\epsilon_0 |\vec{E}|^2 + \frac{1}{2\mu_0}|\vec{B}|^2 \tag{146}$$

Then, using the Maxwell equations, we calculate

$$\begin{aligned}
\frac{\partial \mathcal{E}}{\partial t} &= \epsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} + \frac{1}{\mu_0} \vec{B} \cdot \frac{\partial \vec{B}}{\partial t} \\
&= \frac{1}{\mu_0} \vec{E} \cdot \left(\nabla \wedge \vec{B} - \mu_0 \vec{J}\right) - \frac{1}{\mu_0} \vec{B} \cdot \nabla \wedge \vec{E} \\
&= -\nabla \cdot \left(\frac{1}{\mu_0} \vec{E} \wedge \vec{B}\right) - \vec{E} \cdot \vec{J}
\end{aligned}$$

For the source-free case, we set $\vec{J} = 0$ so that the last term vanishes. Then, this has the form of a conservation equations

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \vec{P} = 0 \tag{147}$$

Remember that $\mathcal{E}$ is the energy density. $\vec{P} \equiv \frac{1}{\mu_0}\vec{E} \wedge \vec{B}$ is called the *Poynting vector* and has the interpretation of the rate of energy flow, or momentum density.

If we now reinstate the sources, this equation has a name:

Poynting's theorem

$$\frac{\partial \mathcal{E}}{\partial t} = -\nabla \cdot \vec{P} - \vec{E} \cdot \vec{J}$$

We integrate Poynting's theorem over a volume V with surface S to obtain an energy balance equation:

$$\underset{(1)}{\frac{d}{dt}\int_V \mathcal{E} dV} = \underset{(2)}{-\int_S \vec{P} \cdot d\vec{S}} - \underset{(3)}{\int_V \vec{E} \cdot \vec{J} dV}$$

where each term has the following interpretation:

1. Rate of increase of electromagnetic energy.
2. Rate of energy flow into V.
3. Rate of work done by the field on sources.

To justify the interpretation of (3), consider a single charge. Remember that work is the product of force and displacement $W = \vec{F} \cdot d\vec{\ell}$, so the rate of work is $\frac{dW}{dt} = \vec{F} \cdot \vec{v}$. For a single charge $\vec{F} = q\vec{E}$ and $q\vec{v}$ is by definition the current $\vec{j}$, so that $\frac{dW}{dt} = \vec{E} \cdot \vec{j}$. This satisfactory interpretation of the third term reinforces the interpretation of $\mathcal{E}$ as energy density.

4.2 Electromagnetic potentials

The introduction of potentials in the time-dependent case is similar to that in the time-independent case. Assume we are interested in a suitable (simply-connected, etc) region of space, then the Maxwell equations imply, first:

$$\nabla \cdot \vec{B} = 0 \Rightarrow \exists \vec{A} \text{ such that } \vec{B} = \nabla \wedge \vec{A};$$

next $\frac{\partial \vec{B}}{\partial t} = \nabla \wedge \frac{\partial \vec{A}}{\partial t}$, so that

$$\nabla \wedge \vec{E} + \frac{\partial \vec{B}}{\partial t} = \nabla \wedge \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0,$$

whence

$$\exists\Phi \text{ such that } \vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla\Phi$$

i.e.

$$\vec{B} = \nabla \wedge \vec{A} \tag{148}$$

$$\vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t} \tag{149}$$

As before, we have the freedom to modify $\vec{A}$ and Φ slightly without changing the electric and magnetic fields. This freedom has the name of *gauge transformation*:

$$\vec{A} \to \vec{A} + \nabla\zeta \tag{150}$$

$$\Phi \to \Phi - \frac{\partial \zeta}{\partial t} \tag{151}$$

This *gauge transformation* and can be exploited, as before, to simplify potentials.

Plugging the expression for the fields in terms of their potentials into the Maxwell equations we obtain

$$\nabla^2\Phi + \frac{\partial}{\partial t}\nabla \cdot \vec{A} = -\frac{1}{\epsilon_0}\rho \tag{152}$$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla\left(\nabla \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}\right) = -\mu_0\vec{J} \tag{153}$$

This is a nice set of equations for the potentials, but they are coupled. You can show that the gauge transformations can be used in order to choose potentials such that

$$\nabla \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0 \tag{154}$$

For this particular choice we obtain

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -\frac{1}{\epsilon_0}\rho \tag{155}$$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0\vec{J} \tag{156}$$

These equations, supplemented with our gauge choice (154) are completely equivalent to Maxwell equations. In order to solve these equations, and give the time-dependent of (26) and (137), we need to introduce a new concept.

4.3 Time-dependent Green's functions

The inhomogeneous wave equations found in the previous section have the structure

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = -4\pi f(\vec{r},t) \tag{157}$$

where c is the velocity and $f(\vec{r},t)$ is a known source term. In order to solve this equation, it is convenient to find the corresponding Green's function, as we did for electrostatics. By definition

$$\nabla^2 G(\vec{r},t;\vec{r}\,',t') - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}G(\vec{r},t;\vec{r}\,',t') = -4\pi\delta(\vec{r}-\vec{r}\,')\delta(t-t') \tag{158}$$

We will focus on situations without boundaries and choose our Green's function to be a function of the differences $\vec{r}-\vec{r}\,'$ and $t-t'$. As we have seen, the delta functions have the following integral representation

$$\delta(\vec{r}-\vec{r}\,')\delta(t-t') = \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')}e^{-i\omega(t-t')} \tag{159}$$

Hence it is natural to represent the Green's function as

$$G(\vec{r},t;\vec{r}\,',t') = \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega g(\vec{k},\omega) e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')}e^{-i\omega(t-t')} \tag{160}$$

Acting on both sides of (160) with the wave operator

$$\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{161}$$

and comparing with the delta function representation (159) we obtain

$$g(\vec{k},\omega) = \frac{1}{4\pi^3}\frac{1}{k^2-\frac{\omega^2}{c^2}} \tag{162}$$

The manipulations we did here are exactly the same we did to arrive to (90), however, there is a crucial difference: as we plug back (162) into (160) and try to do the integral over ω, we find two poles along the real line, at

$$\omega = ck, \quad \omega = -ck \tag{163}$$

Note that this is due to the minus sign in front of the second time derivative in the wave equation. In order to compute (160) we need to give a prescription on how to treat these poles. In similar situations in physics one adds a small imaginary part and moves the poles a little above or below the real axis. In order to know which one of the choices is the correct one, we turn into physical considerations.

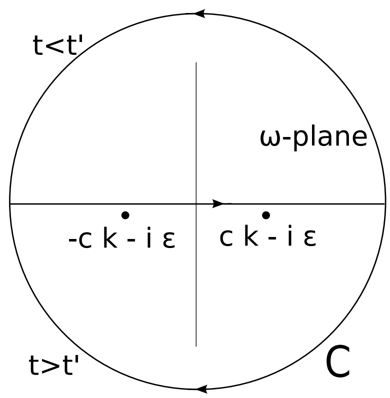
What is the meaning of $G(\vec{r}, t; \vec{r}\,', t')$? it represents the wave caused by a disturbance at $\vec{r} = \vec{r}\,'$, $t = t'$. Hence, for $t < t'$ we demand $G(\vec{r}, t; \vec{r}\,', t')$ to vanish identically (since the disturbance didn't happen yet!)

$$G(\vec{r}, t; \vec{r}\,', t') = 0, \qquad \text{for } t < t' \tag{164}$$

Looking at (160) we see that for $t < t'$ we can compute the integral over ω by closing the contour at infinity from above. Indeed, for $t - t' < 0$ the contribution from the circle above vanishes exponentially. The correct prescription for the poles is then to push them slightly below the real axis:

$$\omega = ck - i\epsilon, \quad \omega = -ck - i\epsilon \tag{165}$$

so that the integral over ω vanishes for the case $t < t'$. See figure



Now, lets turn out to the more interesting situation $t > t'$. In this case we can compute the integral over ω by closing the contour from below. By residue theorem we obtain

$$\int_{\mathcal{C}} d\omega \frac{1}{k^2 - \frac{(\omega + i\epsilon)^2}{c^2}} e^{-i\omega(t-t')} = 2\pi c \frac{\sin\left(ck(t-t')\right)}{k} \tag{166}$$

Hence

$$\begin{aligned} G(\vec{r}, t; \vec{r}\,', t') &= \frac{c}{2\pi^2} \int_{-\infty}^{\infty} d^3k e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')} \frac{\sin\left(ck(t-t')\right)}{k} &(167)\\ &= \frac{2c}{\pi|\vec{r}-\vec{r}\,'|} \int_0^{\infty} dk \sin(k|\vec{r}-\vec{r}\,'|) \sin\left(ck(t-t')\right) &(168)\\ &= \frac{c}{\pi|\vec{r}-\vec{r}\,'|} \int_{-\infty}^{\infty} dk \sin(k|\vec{r}-\vec{r}\,'|) \sin\left(ck(t-t')\right) &(169) \end{aligned}$$

where in the second line we have used spherical coordinates for $\vec{k}$ and integrated over the angular variables. In the third line we have simply extended the integration region to the whole real line. Then, using the exponential form of the trigonometric functions we end up with the integral representation for the delta function (73). One delta is identically zero as $t - t' > 0$, while the other one gives

$$G(\vec{r}, t; \vec{r}', t') = \frac{\delta\left(t' + \frac{|\vec{r}-\vec{r}'|}{c} - t\right)}{|\vec{r} - \vec{r}'|} \tag{170}$$

We see that the Green's function vanishes identically outside the sphere $|\vec{r} - \vec{r}'| = c(t - t')$. This result fits perfectly with the interpretation of $G(\vec{r}, t; \vec{r}', t')$ as a wave that originates from the perturbation at $\vec{r} = \vec{r}'$ and $t = t'$ and then expands spherically at velocity c. This Green's function is also called the retarded Green's function: the effect observed at the point $\vec{r}$ at time t is due to the perturbation originated at a retarded time $t' = t - \frac{|\vec{r}-\vec{r}'|}{c}$. Hence, the final solution to the inhomogeneous wave equation (157) in the absence of boundaries is

$$\psi(\vec{r}, t) = \int \frac{\delta\left(t' + \frac{|\vec{r}-\vec{r}'|}{c} - t\right)}{|\vec{r} - \vec{r}'|} f(\vec{r}', t') dV' dt' = \int \frac{f(\vec{r}', t - \frac{|\vec{r}-\vec{r}'|}{c})}{|\vec{r} - \vec{r}'|} dV' \tag{171}$$

This allows to write the solution for the scalar and vector potential wave equations (155) in the absence of boundaries for arbitrary charge and current distributions.

$$\phi(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}', t - \frac{|\vec{r}-\vec{r}'|}{c})}{|\vec{r} - \vec{r}'|} dV' \tag{172}$$

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}', t - \frac{|\vec{r}-\vec{r}'|}{c})}{|\vec{r} - \vec{r}'|} dV' \tag{173}$$

We leave it as an exercise to show that this solution indeed satisfies the gauge (154).

4.4 Maxwell equations in macroscopic/dielectric media

Consider the basic equations of electrostatics

$$\nabla \wedge \vec{E} = 0, \qquad \nabla \cdot \vec{E} = \frac{1}{\epsilon_0} \rho \tag{174}$$

writing them down for a real situation would require a precise knowledge of the charge density at all points of space. Except some ideal situations (*e.g.* a collection of charges in the vacuum) such a knowledge is impossible. For instance, if we are studying electrostatics in water, on a cubic centimeter there would be billions of atoms with their electrons and protons (which are furthermore vibrating!) and they would have to be accounted for. Luckily one can write an *effective* set of equations which describes electrostatics in macroscopic media. For the simplest case of media they amount to simply replacing the value of ϵ_0 valid for he vacuum by ϵ, which characterizes the medium:

$$\nabla \wedge \vec{E} = 0, \qquad \nabla \cdot (\epsilon\, \vec{E}) = \rho \tag{175}$$

where now the medium itself does not contribute to the charge density. The ratio ϵ/ϵ_0 is called the dielectric constant of the medium, such that the dielectric constant of the vacuum is one. For other materials the dielectric constant is always larger than one. For instance, the dielectric constant of air is very close to 1 (about 1.0005), while the dielectric constant of water is about 2.

These effective equations can be understood as follows. Usually a macroscopic medium contains dipoles [6]. In the absence of an electric field these dipoles are randomly aligned. As we turn an electric field, however, these dipoles align, producing themselves a net electric field. This alignment (and the field the dipoles produce) is proportional to the applied electric field, effectively changing the value of ϵ and giving (175). For most examples studied in this lectures ϵ will be just a constant, for which the effective equations exactly coincide with the equations on the vacuum, upon replacing $\epsilon_0 \to \epsilon$. An exception will be problems where a plane separates two regions with different values of ϵ. In this case, by applying Gauss law, we can see that as we cross the boundary

$$\left(\epsilon_1 \vec{E}_1 - \epsilon_2 \vec{E}_2\right) \cdot \vec{n} = 0 \tag{176}$$

On the other hand, by applying Stokes theorem we obtain

$$\left(\vec{E}_1 - \vec{E}_2\right) \wedge \vec{n} = 0 \tag{177}$$

where $\vec{n}$ is the normal to the boundary. Hence, as we cross the boundary, we must impose continuity for the normal component of $\epsilon\vec{E}$ and the tangential components of $\vec{E}$.

A similar description can be given for the equations of magnetostatics, and the effective equations take the form

$$\nabla \wedge \frac{1}{\mu}\vec{B} = \vec{J}, \qquad \nabla \cdot \vec{B} = 0 \tag{178}$$

The ratio μ/μ_0 is called permeability of the medium. It is one for the vacuum, extremely close to one for air and slightly less than one for water. As before, as we cross the boundary between two regions with different magnetic permeabilities we obtain

$$\left(\frac{1}{\mu_1}\vec{B}_1 - \frac{1}{\mu_2}\vec{B}_2\right) \wedge \vec{n} = 0, \qquad \left(\vec{B}_1 - \vec{B}_2\right) \cdot \vec{n} = 0 \tag{179}$$

Hence, as we cross the boundary the tangential components of $\frac{1}{\mu}\vec{B}$ and the normal components of $\vec{B}$ are continuous. Finally, the effective Maxwell equations in the presence of a medium with constant ϵ, μ take the form

[6] You can think of a dipole as two opposite charges very close to each other. The dipole is said to be aligned in the direction of separation of the charges.

$$
\begin{aligned}
\nabla \wedge \vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0; \qquad \nabla \wedge \vec{B} &= \mu \vec{J} + \mu\epsilon \frac{\partial \vec{E}}{\partial t} \\
\nabla \cdot \vec{B} &= 0, \qquad \nabla \cdot \vec{E} &= \frac{1}{\epsilon}\rho
\end{aligned} \tag{180}
$$

5 Electromagnetic waves

In this section we will study the source-free Maxwell equations and show that they admit wave solutions. We will then study many properties of these solutions.

5.1 Plane waves

We start from the Source-free Maxwell equations:

$$\begin{aligned} \nabla\cdot\vec{B} &= 0, & \nabla\cdot\vec{E} &= 0 \\ \nabla\wedge\vec{E}+\tfrac{\partial\vec{B}}{\partial t} &= 0, & \nabla\wedge\vec{B}-\tfrac{1}{c^2}\tfrac{\partial\vec{E}}{\partial t} &= 0 \end{aligned} \tag{181}$$

we would like to decouple the equations for $\vec{E}$ and $\vec{B}$. Calculate:

$$\begin{aligned} \nabla\wedge(\nabla\wedge\vec{E}) &= \nabla(\nabla\cdot\vec{E})-\nabla^2\vec{E} = -\nabla^2\vec{E} \\ &= -\nabla\wedge\frac{\partial\vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla\wedge\vec{B}) = -\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} \end{aligned}$$

so that

$$\nabla^2\vec{E}-\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0 \tag{182}$$

which is the *wave-equation* with $c=(\epsilon_0\mu_0)^{-1/2}$ as the wave speed. Similarly (exercise)

$$\nabla^2\vec{B}-\frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} = 0 \tag{183}$$

We want to solve these wave equations, so think first about a scalar version

$$\nabla^2F-\frac{1}{c^2}\frac{\partial^2F}{\partial t^2} = \frac{\partial^2F}{\partial x^2}+\frac{\partial^2F}{\partial y^2}+\frac{\partial^2F}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2F}{\partial t^2} = 0 \tag{184}$$

and try $F=f(\vec{k}\cdot\vec{r}-\omega t)$ with constant ω and $\vec{r}$. Then

$$\begin{aligned} \frac{\partial F}{\partial t} &= -\omega f', & \nabla F &= \vec{k}f' \\ \frac{\partial^2F}{\partial t^2} &= \omega^2f'', & \nabla^2F &= k^2f'' \end{aligned}$$

Hence, $F=f(\vec{k}\cdot\vec{r}-\omega t)$ is a solution for any (twice-differentiable) f provided $k^2=\frac{\omega^2}{c^2}$. Note that F is constant on surfaces

$$\vec{k}\cdot\vec{r}-\omega t=\text{const}$$

at a fixed t this is the equation of a plane, so these solutions are called *plane waves.* As time goes forward, the wavefront propagates in the direction of $\vec{k}$. If f is of the form

$$f(\vec{k}\cdot\vec{r}-\omega t)\sim e^{i(\vec{k}\cdot\vec{r}-\omega t)} \tag{185}$$

then they are called *harmonic waves* or or *monochromatic plane waves* with a single frequency ω and wave number k. Note that both the real part and imaginary part of the exponential above will be solutions of the wave equation.

After this discussion, we try

$$\begin{aligned}\vec{E} &= \vec{e}\,E_0\,e^{i(\vec{k}\cdot\vec{r}-\omega t)} \\ \vec{B} &= \vec{b}\,B_0\,e^{i(\vec{k}\cdot\vec{r}-\omega t)},\end{aligned} \tag{186}$$

With the understanding that the electric and magnetic fields are the real part of the solutions. Here $\vec{e}$ and $\vec{b}$ are unit vectors characterizing the direction of the fields and E_0 and B_0 are constants. Plugging these expressions into $\nabla\cdot\vec{B}=0,\ \nabla\cdot\vec{E}=0$ we obtain

$$\vec{e}\cdot\vec{k}=0, \qquad \vec{b}\cdot\vec{k}=0 \tag{187}$$

This means that both $\vec{E}$ and $\vec{B}$ are perpendicular to the direction of propagation of the wave. Such a wave is called transverse wave. The rest of the Maxwell equations imply

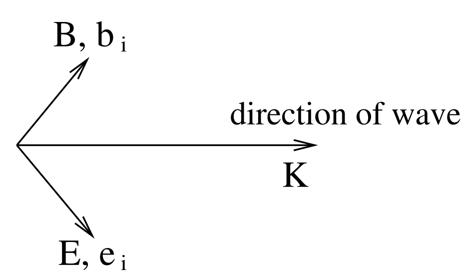
$$(\vec{k}\wedge\vec{e})E_0-\omega\vec{b}B_0=0 \tag{188}$$

$$(\vec{k}\wedge\vec{b})B_0+\frac{\omega}{c^2}\vec{e}E_0=0 \tag{189}$$

This system implies the expected relation $k^2=\frac{\omega^2}{c^2}$, shows that $\vec{e}$ and $\vec{b}$ are perpendicular to each other and fixes the relative magnitudes:

$$B_0=\frac{1}{c}E_0, \qquad \vec{b}=\frac{1}{k}\vec{k}\wedge\vec{e} \quad\to\quad \vec{e}\cdot\vec{b}=0 \tag{190}$$

These are *transverse waves* (unlike *e.g.* sound). The waves travel in the direction of $\vec{k}$, while $\vec{E}$ and $\vec{B}$ lie in wave fronts, transverse to the direction of propagation, and are orthogonal to each other, see figure below.



5.2 Polarization

Let us discuss the phenomenon of polarization. To simplify the expressions, take $\vec{k}$ in the z-direction, $\vec{k} \cdot \vec{r} = kz$. Look in the plane $z = 0$ so that the argument of the wave solutions is simply ωt. Taking linear combinations of the solutions above we can write

$$\vec{E} = \vec{e}_1 \cos \omega t + \vec{e}_2 \sin \omega t \tag{191}$$

$$\vec{B} = \frac{1}{\omega} \vec{k} \wedge \vec{e}_1 \cos \omega t + \frac{1}{\omega} \vec{k} \wedge \vec{e}_2 \sin \omega t \tag{192}$$

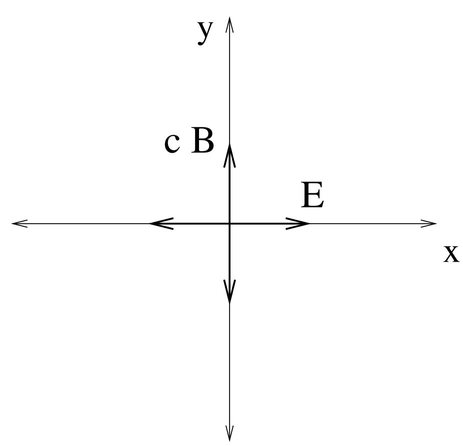
where $\vec{e}_1$ and $\vec{e}_2$ are two a priori independent vectors. To discuss the phenomenon of *polarisation*, we distinguish some particular cases:

(i) $\vec{e}_1, \vec{e}_2$ proportional, say to $\vec{i}$, then

$$\vec{E} = E\vec{i} \cos(\omega t + \delta) \tag{193}$$

$$\vec{B} = \frac{1}{c} E\vec{j} \cos(\omega t + \delta) \tag{194}$$

for some constant δ;



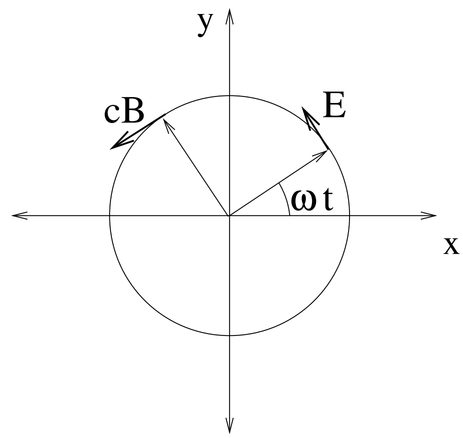
a wave like this is said to be *lineary polarised*; $\vec{E}$ and $\vec{B}$ oscillate parallel to two fixed orthogonal directions.

(ii) $\vec{e}_1, \vec{e}_2$ orthogonal, of equal length, e.g. $\vec{e}_1 = E\vec{i}, \vec{e}_2 = E\vec{j}$, then

$$\vec{E} = E(\vec{i} \cos \omega t + \vec{j} \sin \omega t) \tag{195}$$

$$\vec{B} = \frac{E}{c}(-\vec{i} \sin \omega t + \vec{j} \cos \omega t); \tag{196}$$

a wave like this is said to be *circularly polarised*; $\vec{E}$ and $\vec{B}$ rotate at a constant rate about the direction of propagation.



The direction is anti-clockwise and the wave is *left circularly polarized* if $\vec{e}_1 \wedge \vec{e}_2 \cdot \vec{k} > 0$, and the rotation is clockwise and the wave is *right circularly polarized* if $\vec{e}_1 \wedge \vec{e}_2 \cdot \vec{k} < 0$.

(iii) the general case can be regarded as a combination of waves with different polarizations.

5.3 Reflection and refraction of electromagnetic waves

If you have ever been to a swimming pool you know about the funny properties of light as it crosses the boundary between air and water. The basic laws of optics regarding reflection and refraction of light can be summarized as follows:

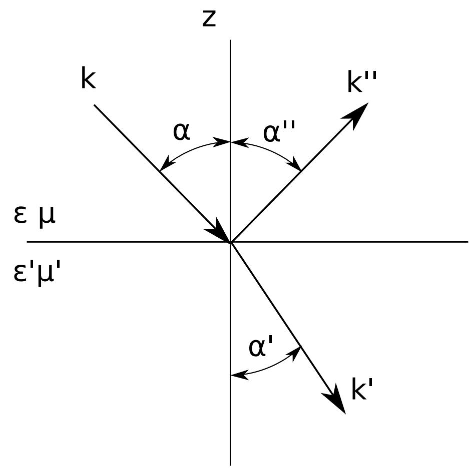
1.- *The angle of reflection equals the angle of incidence.*

2.- *Each material has an index of refraction* n and the angles of incidence α and refraction α' satisfy

$$n \sin\alpha = n' \sin\alpha'$$

For air the index of refraction is almost exactly 1, while for water is about 1.3.

Let us discuss how the laws of reflection and refraction of light at a plane surface between two media of different dielectric properties (for instance air and water) can be deduced from the Maxwell equations. Consider a plane surface located at $z = 0$ separating two media with $\epsilon\mu$ and $\epsilon'\mu'$, and consider an incident wave along $\vec{k}$ which "splits" into a refracted wave along $\vec{k}'$ and a reflected wave along $\vec{k}''$, see figure:



For $z > 0$ the solution corresponds to the incident plus the reflected waves

$$\vec{E} = \vec{E}_0 e^{i(\vec{k}.\vec{r}-\omega t)} + \vec{E}''_0 e^{i(\vec{k}''.\vec{r}-\omega t)} \tag{197}$$

$$\vec{B} = \frac{1}{\omega}\left(\vec{k}\wedge\vec{E}_0 e^{i(\vec{k}.\vec{r}-\omega t)} + \vec{k}''\wedge\vec{E}''_0 e^{i(\vec{k}''.\vec{r}-\omega t)}\right) \tag{198}$$

while for $z < 0$ the solution corresponds to the refracted wave

$$\vec{E} = \vec{E}'_0 e^{i(\vec{k}'.\vec{r}-\omega t)} \tag{199}$$

$$\vec{B} = \frac{1}{\omega}\vec{k}'\wedge\vec{E}_0 e^{i(\vec{k}'.\vec{r}-\omega t)} \tag{200}$$

Here $\vec{E}_0$ and $\vec{k}$ are arbitrary (since we can send in any wave we want) and we need to fix $\vec{E}'_0, \vec{E}''_0$ and $\vec{k}', \vec{k}''$ in terms of these, by requiring the correct boundary conditions at $z = 0$. The wave numbers satisfy

$$k = k'' = \frac{\omega}{v}, \qquad k' = \frac{\omega}{v'} \tag{201}$$

where v and v' are the velocities of the waves in the two media, related to the properties of the medium by

$$v = \frac{1}{\sqrt{\epsilon\,\mu}} = \sqrt{\frac{\epsilon_0\mu_0}{\epsilon\,\mu}}c, \qquad v' = \frac{1}{\sqrt{\epsilon'\mu'}} = \sqrt{\frac{\epsilon_0\mu_0}{\epsilon'\mu'}}c \tag{202}$$

Whatever conditions we have at $z = 0$ the spacial variation of the three waves, as we move along the $z = 0$ plane, should be the same (as they should work for all x, y) hence

$$\vec{k}.\vec{r}|_{z=0} = \vec{k}'.\vec{r}|_{z=0} = \vec{k}''.\vec{r}|_{z=0} \tag{203}$$

independent of the nature of the boundary conditions. In particular the three wave vectors should lie on a plane, let's say (x, z), and their component along the x direction should coincide, namely

$$k \sin\alpha = k' \sin\alpha' = k'' \sin\alpha'' \tag{204}$$

This implies

$$\alpha = \alpha'', \qquad \frac{1}{v}\sin\alpha = \frac{1}{v'}\sin\alpha' \tag{205}$$

These are the two basic laws of optics! the refraction index is usually defined as $n = \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}}$.

We haven't yet imposed the precise boundary conditions at $z = 0$. As we cross the boundary, remember that the normal components of $\epsilon\vec{E}$ and $\vec{B}$ are continuous and the tangential components of $\vec{E}$ and $\mu^{-1}\vec{B}$:

$$\left(\epsilon(\vec{E}_0 + \vec{E}_0'') - \epsilon'\vec{E}_0'\right)\cdot\vec{n} = 0 \tag{206}$$

$$\left(\vec{k}\wedge\vec{E}_0 + \vec{k}''\wedge\vec{E}_0'' - \vec{k}'\wedge\vec{E}_0'\right)\cdot\vec{n} = 0 \tag{207}$$

$$\left(\vec{E}_0 + \vec{E}_0'' - \vec{E}_0'\right)\wedge\vec{n} = 0 \tag{208}$$

$$\left(\frac{1}{\mu}\left(\vec{k}\wedge\vec{E}_0 + \vec{k}''\wedge\vec{E}_0''\right) - \frac{1}{\mu'}\vec{k}'\wedge\vec{E}_0'\right)\wedge\vec{n} = 0 \tag{209}$$

These equations can be used to compute the magnitude of the reflected and refracted fields in terms of the incident one. For instance, assume for simplicity that the electric field is linearly polarized along the direction perpendicular to the incidence plane (the plane formed by $\vec{k}$ and $\vec{n}$), namely $\vec{E}_0, \vec{E}_0', \vec{E}_0''$ are all along the y direction (going out from the picture). In this case, (206) imply

$$E_0 + E_0'' - E_0' = 0 \tag{210}$$

$$\sqrt{\frac{\epsilon}{\mu}}(E_0 - E_0'')\cos\alpha - \sqrt{\frac{\epsilon'}{\mu'}}E_0'\cos\alpha' = 0 \tag{211}$$

Assuming $\mu = \mu'$ (which is almost true for instance for light from air reflected on water) and using (205) we obtain

$$E_0' = \frac{2\cos\alpha\sin\alpha'}{\sin(\alpha+\alpha')}E_0 \tag{212}$$

$$E_0'' = -\frac{\sin(\alpha-\alpha')}{\sin(\alpha+\alpha')}E_0 \tag{213}$$

The correct description of light and the laws of optics was one of the biggest triumphs of electromagnetism!

6 Epilogue: Electromagnetism and Special Relativity

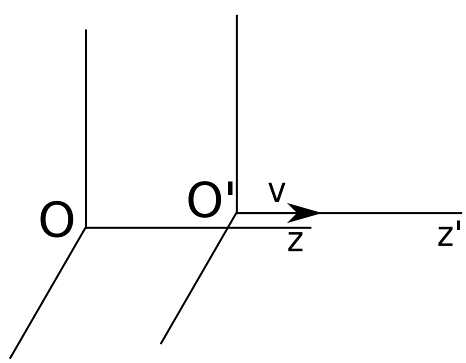
In order to apply the second law of Newton $\vec{F} = m\vec{a}$, compute the path of particles under central forces, etc, you need to choose a system of reference. However, provided the system is inertial you can choose any system you want. In particular, your description and the description of your classmate should be equivalent if your two systems are moving with constant velocity with respect to each other (all inertial frames move with constant velocity with respect to each other). This is also true for other laws of Physics, and so we can make the following postulate:

1.- *The laws of Physics are the same in all inertial frames of reference.*

Should electromagnetism be any different? Imagine it is not. More precisely, if I, in my nice stationary lab, measure forces between wires and charges, I will find out they are described by the electromagnetism equations with some specific ϵ_0 and μ_0. Now, if your lab is moving (at a constant velocity) and you repeat the same experiments, you should find the same results, with the same ϵ_0 and μ_0. In particular, if we both produce electromagnetic waves, both of us will see that these waves move at speed $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$. This leads to the second postulate:

2.- *The speed of light c has the same value in all inertial frames.*

Let us see which consequences can have these two innocent and quite reasonable postulates. Imagine Stella and Leonardo are using two coordinate systems O and O', with space time coordinates (x, y, z, t) and (x', y', z', t'), which are moving at relative speed v along the z component. See figure:



Imagine furthermore that at the common origin of times $t = t' = 0$ the two systems overlap. Now, at $t = t' = 0$ a flash of light is originated at the origin (which is common) and expands in a spherical shape. According to postulate 2 the velocity of the wave in both systems is c, so in you ask Stella what is the shape of this sphere she will tell you:

$$x^2 + y^2 + z^2 - c^2 t^2 = 0 \tag{214}$$

If instead you ask Leonardo what is the shape of the sphere in his system he will tell you:

$$x'^2 + y'^2 + z'^2 - c^2 t'^2 = 0 \tag{215}$$

There should be some way now to translate between the two systems. Galileo will tell you that this is easy, indeed, looking at the pictures:

$$x' = x \tag{216}$$

$$y' = y \tag{217}$$

$$z' = z - vt \tag{218}$$

$$t' = t \tag{219}$$

This is called a Galilean transformation, is the transformation you would use, and has been used for centuries, but it does not agree with (214) and (215)!! If you assume that the relations between the set of variables is linear and you insist on consistency with (214) and (215) you find:

$$x' = x, \quad y' = y \tag{220}$$

$$z' = \frac{z - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{221}$$

$$t' = \frac{t - \frac{v}{c^2} z}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{222}$$

These are called the *Lorentz transformations* and you can see that they reduce to Galilean transformations for velocities much smaller than c. One of the most surprising aspects of these transformations is that the time intervals between the two systems are not the same! and Stella (t) will look at the watch of Leonardo (t') and think that Leonardo's watch is going slower!

Lorentz actually found these transformations by studying the Maxwell equations, and realizing that they were not invariant under Galilean transformations. Einstein proved that these transformations actually followed from his postulates 1 and 2 (which however have wider validity). In this way special relativity was born.