B6.1 (NSDE1) - Problem Sheet 3

Note: Your tutor will inform you which exercises you should solve.

Exercise 1 (Lecture 10)

Let L denote the Lipschitz constant of $\mathbf{f}$. Additionally, let the initial values $\{\mathbf{y}_j\}_{j=0}^{k-1}$ be given. Use the Banach fixed-point theorem (Theorem 2.3 in Lecture 2) to show that, if $h < |a_k|/(L|\beta_k|)$, the multi-step formula

$$\sum_{j=0}^{k} \alpha_j \mathbf{y}_j = h \sum_{j=0}^{k} \beta_j \mathbf{f}(\mathbf{y}_j)$$

has a unique solution $\mathbf{y}_k$.

Exercise 2 (Lecture 10 and 12)

(a) Write the first and second characteristic polynomials of the explicit Euler method, of the implicit Euler method, and of the implicit trapezium rule.

(b) Show that these methods are zero-stable.

(c) Show that the implicit Euler method and implicit trapezium rule are A-stable using the definition of stability domain of multistep methods.

Exercise 3 (Lecture 10)

Let $a, b \in \mathbb{R}$ be some fixed parameters. Show that the multistep methods described by

$$\rho(x) = (x-1)(ax+1-a)\,, \quad \sigma(x) = (x-1)^2 b + (x-1)a + (x+1)/2$$

are of order 2, and show that they are zero-stable if and only if $a \geq 1/2$.

Exercise 4 (Lecture 10)

Show that $hD = -\log(\mathbf{I} - \Delta)(\mathbf{I} - \Delta)E$ and that

$$hD = \left(\Delta - \frac{1}{2}\Delta^2 - \frac{1}{6}\Delta^3 + \dots \right) E\,,$$

and write the formulas of the first and the second characteristic polynomials of the 1-step and 2-step methods associated to this series. Are these methods zero-stable?

Exercise 5 (Lecture 11)

Prove that a linear multi-step method has consistency order p if and only if $\sigma(1) \neq 0$ and

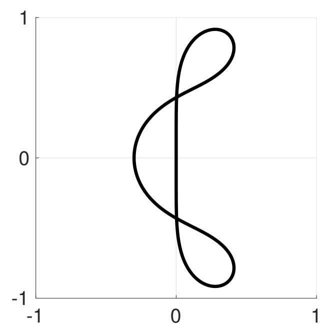
$$\sum_{j=0}^{k} \alpha_j = 0 \quad \text{and} \quad \sum_{j=0}^{k} \alpha_j j^q = q \sum_{j=0}^{k} \beta_j j^{q-1} \quad \text{for} \quad q = 1, \ldots, p\,, \tag{1}$$

and that this condition is equivalent to

$$\rho(e^h) - h\sigma(e^h) = \mathcal{O}(h^{p+1})\,. \tag{2}$$

Exercise 6 (Lecture 11)

The following picture depicts the zero lotus curve of a linear 4-step method.



What can you conclude about the zero-stability and the A-stability of this method?