

B2.1 Introduction to Representation Theory  
Problem Sheet 2, MT 2018

1. Let  $A$  be the three-dimensional  $\mathbb{R}$ -algebra of all upper triangular  $2 \times 2$  matrices over  $\mathbb{R}$ . Find a composition series of the  $A$ -module  $A$ . Show that  $A$  has two simple  $A$ -modules (up to isomorphism), and that one of them occurs twice as a composition factor in your composition series.
2. The **radical**  $\text{rad}V$  of an  $A$ -module  $V$  is defined to be the intersection of all maximal submodules of  $V$ . Let  $A$  be an algebra and consider the  $A$ -modules and  $A$ -submodules  $V \subseteq M_1, M_2 \subseteq X$ .
  - (a) Show that  $M_1/V \cap M_2/V = (M_1 \cap M_2)/V$ .
  - (b) Suppose that  $V$  is finite dimensional. Show that  $V/\text{rad}(V)$  is semi-simple.
  - (c) Show that  $\text{rad}(V)$  is the smallest submodule  $W$  of  $V$  with  $V/W$  semi-simple.
3. Let  $G$  be a finite group and  $N$  a normal subgroup of  $G$ . Let  $V$  be a simple  $KG$ -module. View  $V$  as  $KN$ -module by restriction of the action. Prove that  $V$  as  $KN$ -module is semi-simple.
4. Suppose  $V$  is an  $A$ -module with two composition series, say  $0 \subset U \subset V$  and  $0 \subset W \subset V$  and where  $U \neq W$ .
  - (a) Show that  $V = U \oplus W$  as  $A$ -modules.
  - (b) Now assume that  $U$  and  $W$  are isomorphic, let  $\psi : U \rightarrow W$  be an  $A$ -module isomorphism. For  $\lambda \in K$  fixed, define

$$U_\lambda := \{u + \lambda\psi(u) \mid u \in U\}.$$

Check that  $U_\lambda$  is a submodule of  $V$  and that it is isomorphic to  $U$ .

- (c) Deduce that  $V$  has infinitely many composition series when  $K$  is infinite.
5. Let  $A = \mathbb{C}G$  be the group algebra of the dihedral group of order 10,

$$G = D_{10} = \langle \sigma, \tau : \sigma^5 = 1, \tau^2 = 1, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle.$$

Suppose  $\zeta$  is a 5-th root of 1 (and  $\zeta \neq 1$ ). You may assume that the matrices

$$\rho(\sigma) = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, \quad \rho(\tau) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

satisfy the defining relations for  $G$ , hence give a group homomorphism  $\rho : G \rightarrow GL_2(\mathbb{C})$ .

- (a) Prove that the representation  $\rho$  is irreducible, (that is the  $A$ -module  $V$  corresponding to the representation  $\rho$  is simple).

- (b) Suppose  $G$  is any finite group, and  $\rho_1, \rho_2 : G \rightarrow GL_n(\mathbb{C})$  are representations. Show that if  $\rho_1, \rho_2$  are equivalent then for all  $g \in G$ , we have  $\text{tr}(\rho_1(g)) = \text{tr}(\rho_2(g))$  where  $\text{tr}(X)$  is the usual trace of a matrix  $X$ .
  - (c) Deduce that if  $G = D_{10}$ , then  $G$  has at least two non-equivalent irreducible representations of degree two (equivalently two non-isomorphic two-dimensional simple  $\mathbb{C}G$ -modules).
6. Let  $A$  be a finite-dimensional algebra. An ideal  $I$  of  $A$  is called *nilpotent* if there is some natural number  $n \geq 1$  with  $I^n = 0$ , that is, such that  $x_1 \cdots x_n = 0$  for all  $x_i \in I$ . Define the *radical* of the algebra  $A$  as

$$\text{rad}(A) = \{a \in A \mid a \cdot S = 0 \text{ for any simple } A\text{-module } S\}.$$

- (a) Show that the sum of two nilpotent ideals is nilpotent.
- (b) Show that  $\text{rad}(A)$  is a (two-sided) ideal in  $A$ .
- (c) By considering a composition series of  $A$ , or otherwise, show that  $\text{rad}(A)$  is nilpotent.

Conclude that the radical of an algebra  $A$  coincides with the largest nilpotent ideal of  $A$ .

7. (a) Show that the only one-dimensional  $\mathbb{C}S_n$ -modules are the trivial module and the sign module. (The latter is the module in which each permutation acts by its signature.)
- (b) Determine all the simple  $\mathbb{C}S_3$ -modules, up to isomorphism.
- (c) A group representation  $\rho : G \rightarrow GL(V)$  is called *faithful* if

$$\ker \rho = \{1\}.$$

Determine all the irreducible *non-faithful* representations of  $S_n$  (up to equivalence).