B5.6: Nonlinear Systems-Sheet 2 (solutions)

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Q1 (heteroclinic orbit) For the system

$$\dot{x} = x - y, \quad (1)$$
$$\dot{y} = -\alpha x + \alpha xy, \quad (2)$$

we note that differentiating $I(t) = (y - 2x + x^2)e^{-2t}$ gives the result:

$$\dot{I}(t) = [(2-\alpha)x + (\alpha - 2)xy]e^{-2t},$$

which suggests that $\alpha = 2$ to ensure that $\dot{I} = 0$.

The fixed points of (1)-(2) are $(0,0)$ and $(1,1)$, regardless of the value for α. For $\alpha = 2$, we note that $I = 0$ for both fixed points, meaning that the orbit connecting the two are a branch of a level set in I. Using linearization, $(0,0)$ has eigenvalues $\lambda = \{-1, 2\}$, so that it is a saddle point, whilst for $(1,1)$, $\lambda = \{1, 2\}$, which makes it unstable. As such, a heteroclinic orbit connects the two fixed points given that, on this level set, $(x; y) \to (0,0)$ as $t \to \infty$, whilst $(x, y) \to (1,1)$ as $t \to -\infty$. Furthermore, by setting $I = 0$, we find the trajectory in the $(x; y)$ plane is given by:

$$y = 2x - x^2. \quad (3)$$

To find the closed form of the heteroclinic orbit, we substitute (3) into (1) to obtain:

$$\dot{x} = x(x-1) \quad \Longrightarrow \quad x(t) = \frac{1}{1 - B\exp(t)},$$

where B is an integration constant. Using (3) we obtain the closed form for $y(t)$:

$$y(t) = \frac{2}{1 - B\exp(t)} - \left(\frac{1}{1 - B\exp(t)}\right)^2.$$

Note that (1)-(2) tend to $(1,1)$ as $t \to -\infty$ and $(0,0)$ as $t \to +\infty$. To find B, we need to specify an initial point on the branch which is not either of the fixed points.

Q2 (simple pendulum) Consider the nondimensionalized pendulum

$$\ddot{x} + \sin x = 0. \tag{4}$$

By multiplying (4) through by $\dot{x}$ and integrating in time, we note that:

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2}\dot{x}^2 + V(x)\right) &= 0, \qquad (5)\\ \implies \frac{1}{2}\dot{x}^2 - \cos(x) &= E_m, \quad V(x) = -\cos(x), \qquad (6) \end{aligned}$$

where E_m is an integration constant analogous to the total mechanical energy of the system and $V(x)$ is the potential. From the potential, we have fixed points at $x = 2n\pi$ (pendulum is pointing down) and $x = (2n-1)\pi$ (pendulum is pointing up) for $n \in \mathrm{Z} \cup 0$, with corresponding potential values $V = \{-1, 1\}$, respectively. The formers are centers whilst the latter are saddle points. Perturbing an upright pendulum at rest, say at $x = \pi$, results in the system tending towards another fixed point with V = 1. In this case, depending on the direction of perturbation, this will be either $x = -\pi$ or 3π. The orbit connecting the two saddle points is a heteroclinic orbit as the system will tend to one fixed point for $t \to \infty$, and the other as $t \to -\infty$.

Result (6) suggests that the trajectories form level sets in E_m. Therefore, orbits with $E_m \leq 1$ (i.e. those contained within the symmetric pair of heteroclinic orbits or, alternatively, are confined to the potential well) form an invariant set. However, the trajectories do not form an attracting set, because flows in the neighborhood of this invariant set will not tend to these orbits for any other value of E_m. The phase portrait of the system is shown in Fig. 1.

Q3 (linearisation) **1)** Studying the system

$$\begin{aligned} \dot{x} &= 2x - 2xy, \qquad (7)\\ \dot{y} &= 2y - x^2 + y^2, \qquad (8) \end{aligned}$$

we note that there are four fixed points:$(x, y) = \{(0, -2), (0, 0), (-\sqrt{3}, 1), (\sqrt{3}, 1)\}$ whilst the system has a Jacobian given by:

$$Df(x, y) = \begin{bmatrix} 2 - 2y & -2x \\ -2x & 2 + 2y \end{bmatrix}$$

- For $(0, -2)$, the eigenvalues by linearization are $\lambda = \{-2, 6\}$, with corresponding eigenvectors $v^{(1)} = (0, 1)^T$ and $v^{(2)} = (1, 0)^T$. The fixed point is a saddle node.

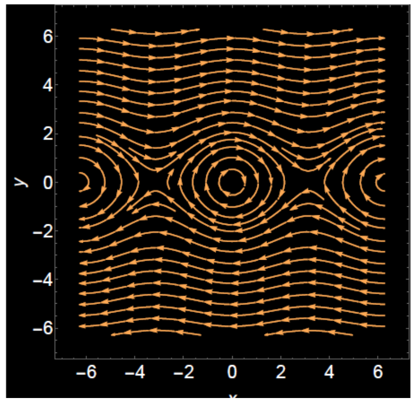

Figure 1: Phase portrait of (4) for $\dot{x} = y$ and $\dot{y} = -\sin(x)$.

- For $(0,0)$, the eigenvalues are $\lambda = \{2,2\}$, with eigenvectors $v^{(1)} = (0,1)^T$ and $v^{(2)} = (1,0)^T$. The fixed point is an unstable star node.
- For $(-\sqrt{3},1)$ we have $\lambda = \{-2,6\}$ with $v^{(1)} = (-\sqrt{3},1)^T$ and $v^{(2)} = (1/\sqrt{3},1)^T$. The fixed point is a saddle node.
- For $(\sqrt{3},1)$ we have $\lambda = \{-2,6\}$ with $v^{(1)} = (\sqrt{3},1)^T$ and $v^{(2)} = (-1/\sqrt{3},1)^T$. The fixed point is a saddle node.

System (7)-(8) has three-fold symmetry, as is evident in Fig. 2(a).

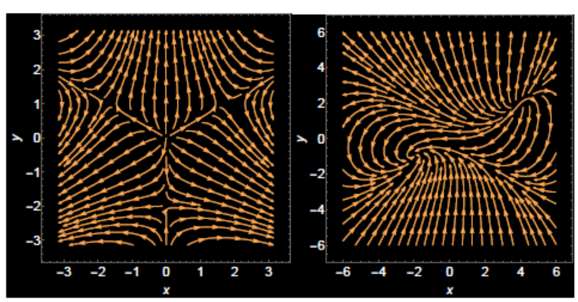

Figure 2: Left: Phase portrait of (7)-(8) Right: Phase portrait of (9)-(10).

2) We now look at the system

$$\dot{x} = -4y + 2xy - 8, \quad (9)$$
$$\dot{y} = -x^2 + 4y^2, \quad (10)$$

which has fixed points $(x, y) = \{(-2, -1), (4, 2)\}$ and a Jacobian given by:

$$Df(x,y) = \begin{bmatrix} 2y & 2x-4 \\ -2x & 8y \end{bmatrix}.$$

- $(-2, -1)$ has eigenvalues $\lambda = \{-5 - i\sqrt{23}, -5 + i\sqrt{23}\}$ resulting in a stable spiral.
- $(4, 2)$ has eigenvalues $\lambda = \{8, 12\}$ with $v^{(1)} = (1, 1)^T$ and $v^{(2)} = (1, 2)^T$ giving an unstable node.

Phase portrait of system (9)-(10) is shown in Fig. 2(b).

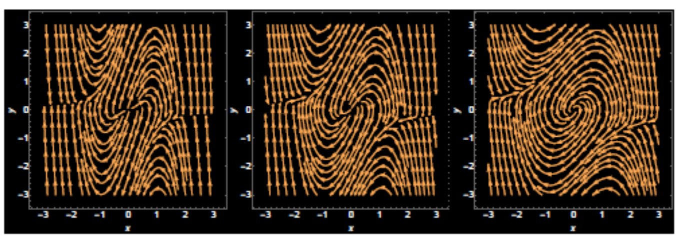

Figure 3: Phase portrait of (11) for (a) $\varepsilon = 3$, (b) $\varepsilon = 2$, and (c) $\varepsilon = 1$.

3) Lastly, we have the Van der Pol oscillator which, using $\dot{x} = y$, can be written as:

$$\dot{x} = y, \quad \dot{y} = -\varepsilon(x^2 - 1)y - x. \quad (11)$$

System (11) has a single fixed point at $(0, 0)$ with Jacobian:

$$Df(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & \varepsilon \end{bmatrix},$$

and the characteristic polynomial:

$$0 = \lambda^2 - \varepsilon\lambda + 1,$$
$$\implies \lambda = \frac{1}{2}\left(\varepsilon \pm \sqrt{\varepsilon^2 - 4}\right).$$

We note 5 important regimes:

- $\varepsilon > 2$, where λ has two real solutions which have the limit $\lambda \sim \{0, \varepsilon\}$ as $\varepsilon \to \infty$. (Unstable node)
- $\varepsilon = 2$, where $\lambda = 1$ is doubly degenerate. (Unstable degenerate node as $v^{(1)} = (1,1)^T$ and $v^{(2)} = (0,0)^T$)
- $0 < \varepsilon < 2$, where λ is imaginary and has positive real part. (Unstable spiral)
- $\varepsilon = 0$, where λ is purely imaginary. (Center)
- $-2 < \varepsilon < 0$, where λ is imaginary and has negative real part. (Stable spiral)
- $\varepsilon = -2$, where $\lambda = -1$ is doubly degenerate. (Stable degenerate node as as $v^{(1)} = (-1,1)^T$ and $v^{(2)} = (0,0)^T$)
- $\varepsilon < -2$, where λ has two real solutions which have the limit $\lambda \sim \{\varepsilon, 0\}$ as $\varepsilon \to -\infty$. (Stable node)

Phase-plane portrait for (11) with $\varepsilon = \{3, 2, 1\}$ is shown in Fig. 3.

Q4 We consider a vector field $\dot{x} = f(x)$ in $\mathbb{R}^n$ with a fixed point x_0 and first integral $H(x)$. Let x_0 be a nondegenerate minimum of $H(x)$. We know that $\dot{H}(x) = 0$, by definition of the first integral, and $\nabla H(x_0) = 0$ by definition of the minimum. If x_0 is a nondegenerate minimum, then:

$$H(x) > H(x_0).$$

Let us define the difference in H as:

$$\Delta H(x) = H(x) - H(x_0).$$

We realize that $\Delta H(x_0) = 0$ and $\Delta \dot{H}(x_0) = 0$, meaning that we can use $\Delta H(x)$ as a Lyapunov function and subsequently show x_0 is a stable fixed point by Lyapunov's method. Note that x_0 is not asymptotically stable because a small neighborhood of points near x_0 will not tend to it as $t \to \infty$, because $H(x)$ forms level sets to which trajectories are confined. As such, x_0 is only Lyapunov stable, given that trajectories which begin within a local neighborhood of this fixed point will stay there for all time.

Q5 (A system with a non-hyperbolic fixed point) We study the system given by:

$$\dot{x} = y - x^3, \quad (12)$$
$$\dot{y} = -x^3, \quad (13)$$

which has a fixed point at the origin.

By linearisation around the fixed point, we find:

$$Df(0,0) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

with corresponding eigenvalues given by $\lambda = \{0, 0\}$, which tells us nothing about whether the origin is stable or not in (12)-(13). Instead, we introduce a Lyapunov function $Q(x, y)$ of the form:

$$Q(x, y) = x^m + cy^n, \quad (14)$$

which has the required properties $Q(0, 0) = 0$ and $Q(x, y) > 0, \forall(x, y) \neq (0, 0)$, provided that m and n are even. Differentiating (14) with respect to t, we find:

$$\dot{Q} = mx^{m-1}y - mx^{m+2} - cnx^3y^{n-1},$$

which suggests that $m = 4, n = 2$ and $c = 2$ by matching the powers of x and y and the coefficients of the first and third term in (53). We note that: $\dot{Q} = -4x^6$ which is less than zero for all points except those with $x = 0$. As such, we have found a Lyapunov function:

$$Q(x, y) = x^4 + 2y^2,$$

which shows that $(0, 0)$ is Lyapunov stable.

To show that the origin is asymptotically stable, note that, although Q vanishes along the line $x = 0$ according to (54), the trajectory is knocked off of this line according to (48), provided $y \neq 0$. For all other points not on $x = 0$, $\dot{Q} < 0$ so that the trajectory spirals into the origin as $t \to \infty$.