2. Nonlinear Systems

2.1 Existence/Uniqueness

$$\dot{x} = f(x) \qquad f: E \subset \mathbb{R}^n \to \mathbb{R}^n$$

autonomous $\partial_t f = 0$

Thm. $E \subset \mathbb{R}^n$, $f: E \to \mathbb{R}^n$ $f \in C^1(E)$
$x_0 \in E \Rightarrow \exists c$ s.t.
$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$ has 1! soln $x(t)$ on $[-c, c]$

- E: differentiable m'fld (generalizato)
- f Lipschitz
- local thm.

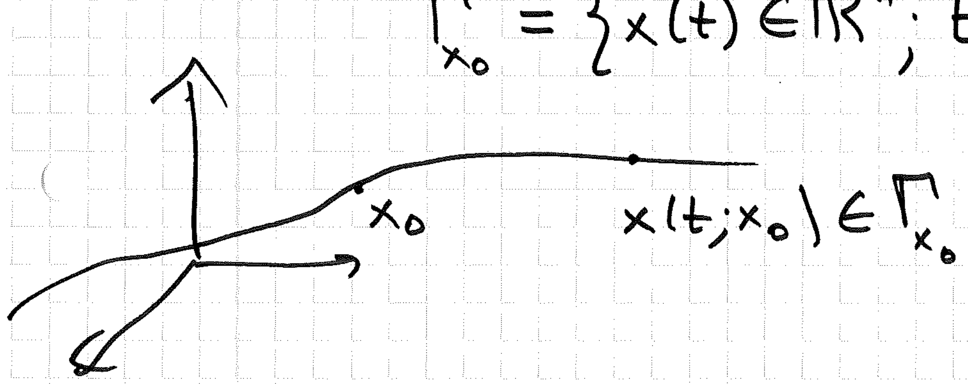
Note: If $x(t)$ is a soln $\Rightarrow \forall \tau \in \mathbb{R}$, $x(t+\tau)$ is also a soln (consequence of $\partial_t f = 0$).

From now on, we assume that $\mathbb{R}$ is the maximum interval of existence (Global)

2.2 Flows, asymptotic & invariant sets

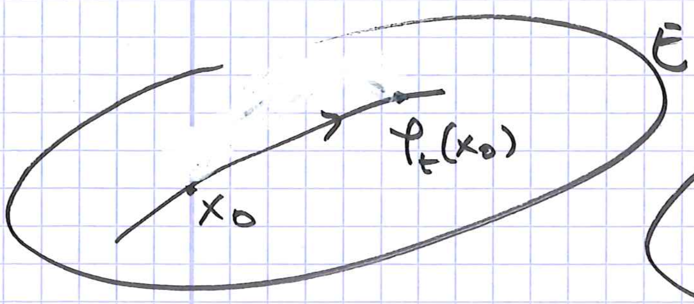
The phase space is $\mathbb{R}^n$. An orbit based on x_0 is the curve $\Gamma_{x_0} \subset \mathbb{R}^n$:

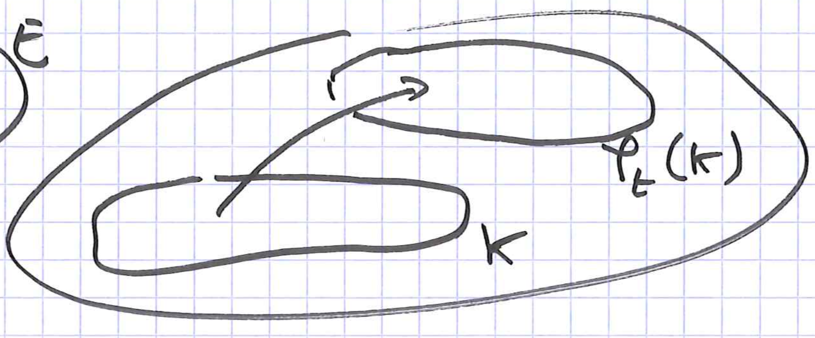
$$\Gamma_{x_0} = \{x(t) \in \mathbb{R}^n ; t \in \mathbb{R}, x(t_0) = x_0, \dot{x} = f(x)\}$$



The <u>flow</u> is the map $\varphi_t : E \to E$ s.t

$$\varphi_t(x_0) = x(t; x_0) \quad \forall x_0 \in K \subset E$$





<u>Properties of flows</u>

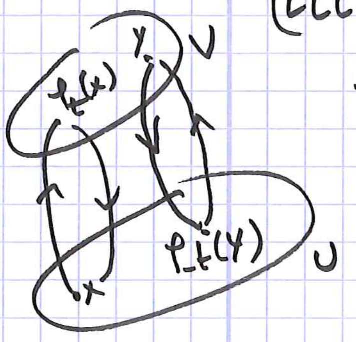
(i) $\varphi_0(x_0) = x_0 \quad \varphi_0 = id.$

(ii) $\varphi_{s+t} = \varphi_s \circ \varphi_t = \varphi_s \circ \varphi_t$

(iii) x_0, U nbhd of x_0, $V = \varphi_t(U)$

$$\Rightarrow \varphi_{-t}(\varphi_t(x)) = x \quad \forall x \in U$$

$$\varphi_t(\varphi_{-t}(y)) = y \quad \forall y \in V$$



<u>Invariant sets</u> $f \in C^1(E)$, $\varphi_t : E \to E$

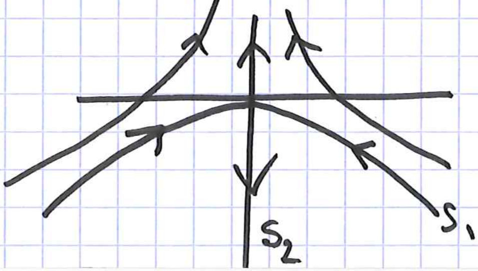
Then, $S \subset E$ is an invariant set of φ_t if

$$\varphi_t(S) \subset S \quad \forall t \in \mathbb{R}$$

<u>Example</u> $\begin{cases} \dot{x}_1 = -x_1 \\ \dot{x}_2 = x_2 + x_1^2 \end{cases} \Rightarrow \varphi_t(x_0) = \begin{bmatrix} x_{0,1} e^{-t} \\ x_{0,2} e^{t} + \frac{x_{0,1}^2}{3}(e^{t} - e^{-2t}) \end{bmatrix}$

Then $S_1 = \{x \in \mathbb{R}^2 \mid x_2 = -x_1^2/3\}$ is invariant

So is $S_2 = \{x \in \mathbb{R}^2 \mid x_1 = 0\}$



Asymptotic sets

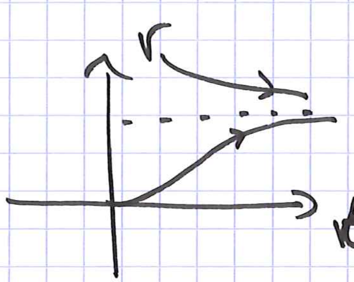
Def. A pt $p \in E$ is an ω-limit pt of φ_t if $\exists$ a sequence $t_1 < t_2 < \dots < t_n$, $t_i \xrightarrow[i\to\infty]{} \infty$ s.t. $\lim_{i\to\infty} \varphi_{t_i}(x) = p$

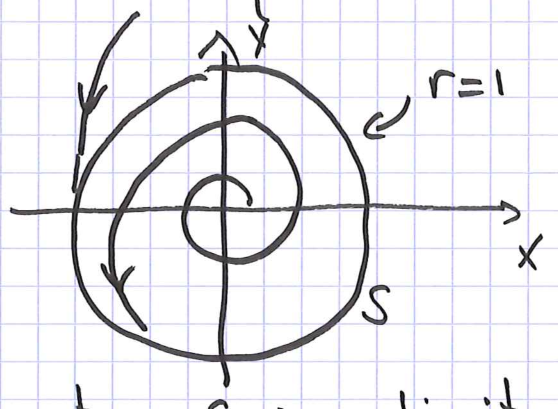
$t_i \to -\infty$ $\quad t_1 > t_2 > \dots > t_n$ $\quad$ α-limit pt

Example

$$\begin{cases} \dot{x} = -y + x(1-x^2-y^2) \\ \dot{y} = x + y(1-x^2-y^2) \end{cases}$$

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right| \Rightarrow \begin{cases} \dot{r} = r(1-r^2) \\ \dot{\theta} = 1 \end{cases}$$

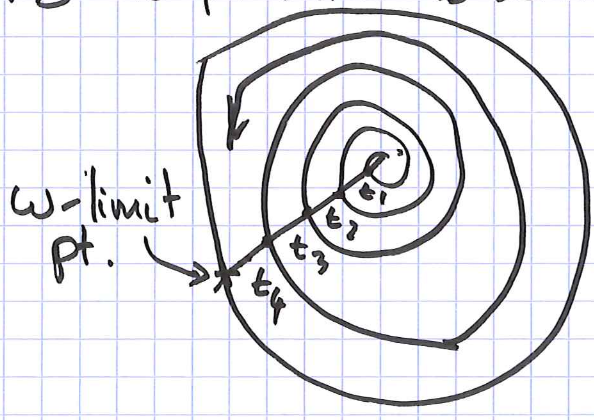




$r=1$ ω-limit set.

$r=0$ α-limit set.

NB no pt on S is a limit pt of the flow

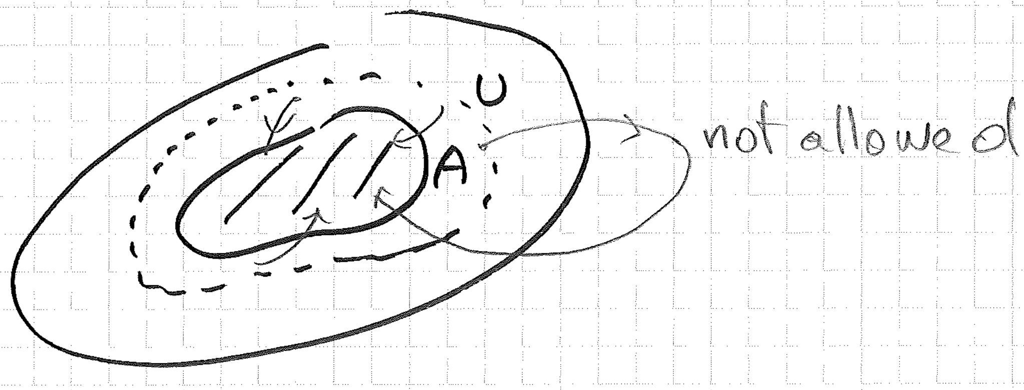


every pt on S is an ω-limit pt.

. Attracting set

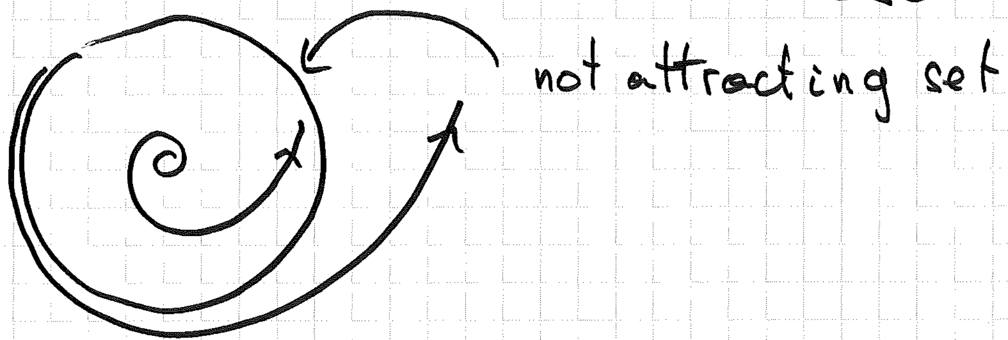
Def: An attracting set is a closed invariant set A s.t $\exists$ a nbhd U of A s.t:

$$\begin{cases} \varphi_t(x) \in U & \forall t \geq 0 \\ \varphi_t(x) \xrightarrow[t\to\infty]{} A & \forall x \in U \end{cases}$$



The domain of attraction $D(A) = \bigcup_{t \leq 0} \varphi_t(U)$

Ex:



$D(A) = \mathbb{R}^2 \setminus \{0\}$

Attracting set and attractor

Def: A dense orbit $\tilde{\Gamma} \in A$ is s.t $\forall \epsilon > 0$
$\forall \Gamma \in A$, $\exists\, x, \tilde{x} \in \Gamma, \tilde{\Gamma}$ with $|x - \tilde{x}| < \epsilon$

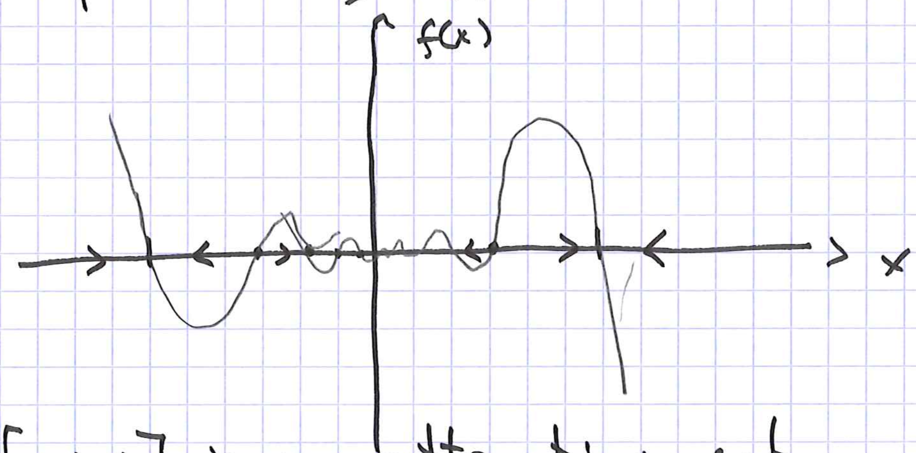
Dense orbit goes as close as wanted to any pt of A.

An attracting set with a dense orbit is an attractor.

Example:

$$\dot{x} = \begin{cases} -x^4 (\sin \pi/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

fixed pts $x = 0, \pm 1/n$



$A = [-1, 1]$ is an attracting set

$\exists$ a nbhd U of A $U = [-1-\epsilon, 1+\epsilon]$ $\epsilon > 0$

$\varphi_t(U) \subset U \ \forall t \geq 0$

$\varphi_t(x) \underset{t \to \infty}{\longrightarrow} A$

Each fixed pt $x = \pm \frac{1}{(2n-1)}$ $n \in \mathbb{N}$ is an attractor

But $x = 0$ is not an attracting set.

2.3 Stability

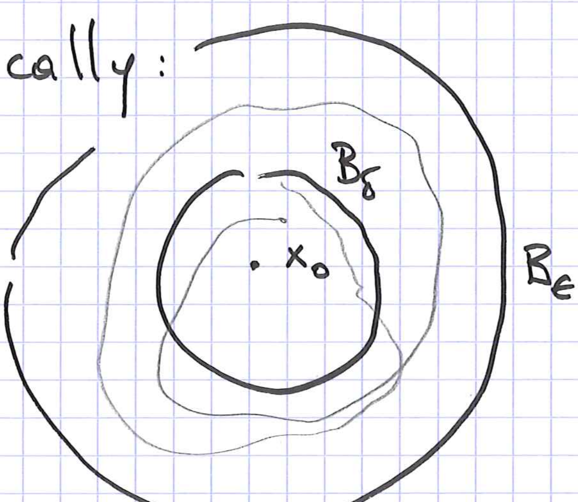
$\dot{x} = f(x)$, $x \in E \subseteq \mathbb{R}^n$ $\quad f \in C^1(\mathbb{R}^n)$

x_0 fixed pt : $f(x_0) = 0$

"x_0 is stable if a soln close to x_0, remains close to x_0 $\forall t$"

Def. The fixed pt x_0 is (Lyapunov) stable if $\forall \epsilon > 0$, $\exists \delta > 0$ s.t $\forall x \in B_\delta(x_0)$ and $t \geq 0$, $\varphi_t(x_0) \in B_\epsilon(x_0)$

Geometrically:

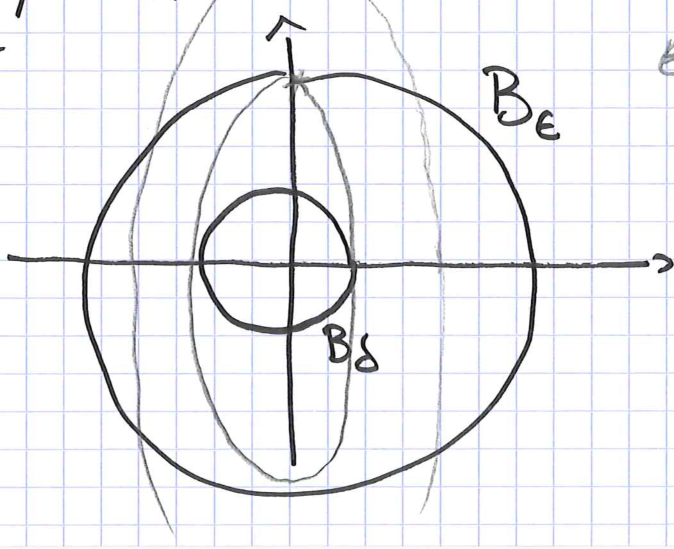


$B_\delta(x_0)$ closed ball of radius δ around x_0

$$B_\delta(x_0) = \{ x \in \mathbb{R} \mid \|x - x_0\| \leq \delta \}$$

x_0 is unstable (Lyapunov) if it is not stable.

Ex: $\begin{cases} \dot{x} = y \\ \dot{y} = -4x \end{cases} \Rightarrow 4x^2 + y^2 = C$



$\epsilon^2 = C = 4\delta^2$

$\Rightarrow \delta = \frac{\epsilon}{2}$

Def x_0 is asymtotically stable if

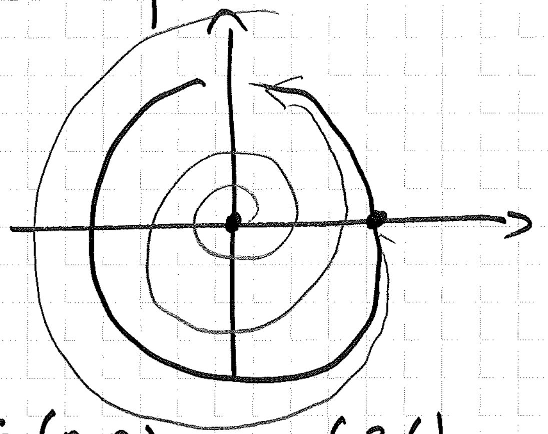
1/ x_0 is stable

2/ $\exists\, \delta > 0$ s.t $\forall\, x \in B_\delta(x_0)$, $\varphi_t(x) \xrightarrow[t\to\infty]{} x_0$

Stronger notion than stability.

Why do we need 1/ ?

Consider $\begin{cases} \dot r = r(1-r) \\ \dot\theta = \sin^2 \theta/2 \end{cases}$

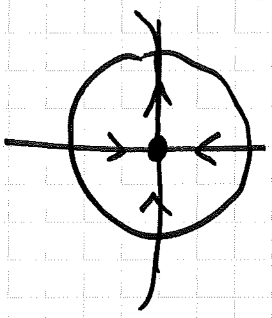


(0,0) unstable

(1,0) is the ω-pt $\forall\, x \notin (0,0)$ (2/)

But (1,0) not stable (not 1/)

Locally



not stable.

For linear systems w/ constant coeff. semi-simple

$\dot x = Ax$, $x(t_0) = x_0$

$x = 0$ is as. stable if $\mathrm{Re}(\lambda) < 0$ $\forall\, \lambda \in \mathrm{Spec}(A)$

is stable if $\mathrm{Re}(\lambda) = 0$ $\forall\, \lambda$

⚠ $x = 0$ is not necessarily stable if A is not semi-simple.

For nonlinear systems

Testing stability may be difficult.

1/ Lyapunov functions

$\dot{x} = f(x) \qquad x_0 : f(x_0) = 0$

$V : W \to \mathbb{R} \qquad W \subseteq \mathbb{R} \qquad V \in C^1(W)$

V is a **Lyapunov fn** if $\exists V \in C^1(W)$, W a nbhd of x_0

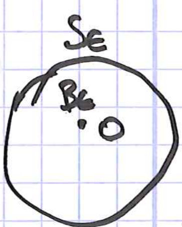
s.t. i/ $V(x_0) = 0$, $V(x) > 0 \ \forall x \neq x_0 \ x \in W$

ii/ $\dot{V}(x) \leq 0 \ \forall x \in W \setminus \{x_0\}$

Thm (a) if V is a Lyapunov fn of f in a nbhd of x_0 then x_0 is stable.

(b) if, furthermore, $\dot{V}(x) < 0 \ \forall x \in W \setminus \{x_0\}$ then x_0 is as. stable

Pf (Part (a)) Let $x_0 = 0$, $\epsilon > 0$ s.t. $\overline{B}_\epsilon(0) \subset E$

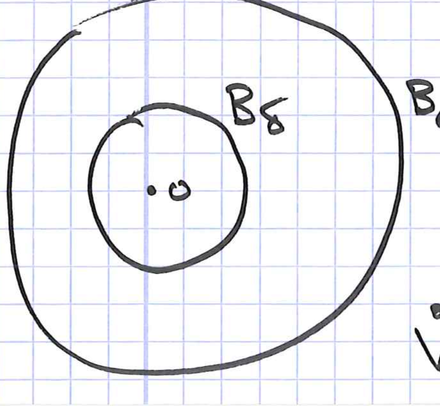


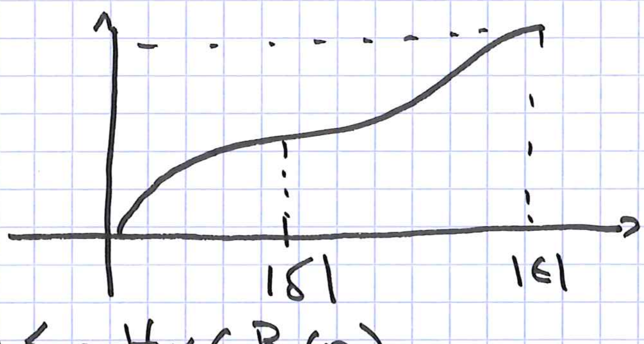
Let S_ϵ be the closure of $B_\epsilon(0)$

m_ϵ the min of $V(x)$ on the boundary of S_ϵ

We have $m_\epsilon > 0$ since $V(x) > 0 \ \forall x \neq 0$.

$V(x)$ is continuous and $V(0) = 0 \Rightarrow \exists \delta > 0$ s.t. $\forall x \in B_\delta(0)$ $V(x) < m_\epsilon$





$\dot{V}(x) \leq 0 \ \forall x \in B_\epsilon(0)$

$\Rightarrow \forall x_0 \in B_\delta(0),\ t>0 \quad V(\varphi_t(x_0)) \leq V(x_\delta) < m_\epsilon$

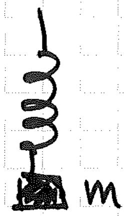
This implies that $\varphi_t(x_0) \subset B_\epsilon(0)$. Indeed assume by contradiction that $\exists\, t_1 \in \mathbb{R}$ and $x_0 \in B_\delta(0)$ s.t $\varphi_{t_1}(x_0) \in S_\epsilon \Rightarrow V(\varphi_{t_1}(x_0)) \geq m_\epsilon$ (a contradiction)

$\Rightarrow \quad \forall \epsilon>0\ \exists\, \delta>0$ s.t $|\varphi_t(x_0)| < \epsilon \quad \forall x \in B_\delta(0)$
$\forall t \geq 0$

$\Leftrightarrow$ Stability.

Example: damped spring:

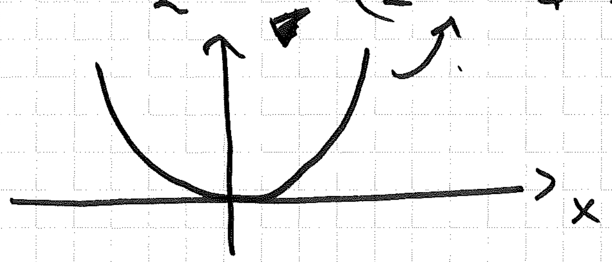
$$m\ddot{x} + k(x+x^3) + \alpha\dot{x} = 0$$

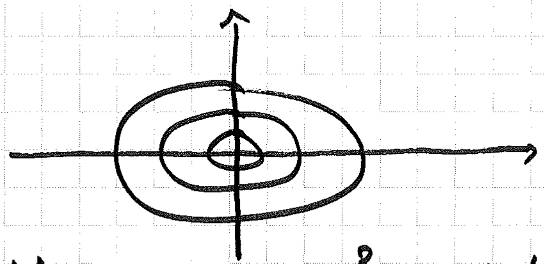


nonlinear spring.

$\alpha > 0$ damping.

1/ $\alpha = 0$ $E = \frac{m\dot{x}^2}{2} + k\left(\frac{x^2}{2} + \frac{x^4}{4}\right)$ energy





Take $V = \frac{m y^2}{2} + k\left(\frac{x^2}{2} + \frac{x^4}{4}\right)$ Lyap. fn for $(x,y) = (0,0)$

1/ $V(0) = 0$, $V(x,y) > 0$ $\forall (x,y) \neq (0,0)$

2/ $\dot{V} = 0$

$\Rightarrow$ $(0,0)$ is stable (but not as. stable)

2/ $\alpha \neq 0$ E is <u>not</u> a Lyapunov fn. (for as. stab.)

indeed $\dot{E} = -\alpha m y^2$ and $\dot{E} \not< 0$ $\forall y$

Try $V = E + \beta\left(xy + \alpha \frac{x^2}{2}\right)$

$$\Rightarrow \dot{V} = \dot{E} + \beta \frac{d}{dt}\left(xy + \alpha\frac{x^2}{2}\right)$$

$$= -\beta \frac{k}{m}(x^2 + x^4) - \alpha(m - \beta) y^2$$

$\Rightarrow$ If β is small enough $V > 0$ $\dot{V} < 0$ $\forall (x,y) \neq (0,0)$

$\Rightarrow$ $(0,0)$ is as. stable (Globally since $W = \mathbb{R}^2$)