B5.6: Nonlinear Systems-Sheet 2

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Q1 (A heteroclinic orbit) A heteroclinic orbit is an orbit that connects two fixed points. Find the value of α such that the system

$$\dot{x} = x - y, \tag{1}$$
$$\dot{y} = -\alpha x + \alpha x y \tag{2}$$

admits the first integral $I = (y - 2x + x^2)e^{-2t}$. (A scalar function $I(x,t)$ is a *first integral* if $\dot{I} = 0$ on all trajectories.) Compute the fixed points and show that a branch of the level set of this first integral is a heteroclinic orbit. Can you find a closed form solution of this orbit?

Q2 (Simple pendulum) The equation for the simple pendulum is

$$\ddot{x} + \sin(x) = 0, \quad x \in \mathbb{R}\,.$$

Find the potential for this system and use it to identify important orbits. In particular identify the fixed points and show that there exist heteroclinic orbits for this system. Sketch the phase portrait. Show that the orbits contained within a symmetric pair of heteroclinic orbits (called a *heteroclinic cycle*) form an invariant set. Is this an attracting set? (This equation can be solved in terms of Elliptic integrals of the first kind, but in this case you are asked to answer all these questions without solving the equation explicitly).

Q3 (Linearisation) Consider the systems below. Find the fixed points and determine their stability through linearisation whenever possible. For the last system describe stability with respect to the parameter.

$$\dot{x} = 2x - 2xy, \tag{3}$$
$$\dot{y} = 2y - x^2 + y^2 \tag{4}$$

$$\dot{x} = -4y + 2xy - 8 \tag{5}$$
$$\dot{y} = -x^2 + 4y^2 \tag{6}$$

$$\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = 0. \tag{7}$$

Q4 Consider a vector field $x = f(x), x \in \mathbb{R}^n$. Assume that $H = H(x)$ is a first integral ($\dot{H} = 0$). Let x_0 be a fixed point. Prove that it if x_0 is a nondegenerate minimum of H, then x_0 is stable.

Q5 (A system with a non-hyperbolic fixed point) Show that the origin is an asymptotically stable point of equilibrium for the nonlinear system

$$\begin{aligned} \dot{x} &= y - x^3, \\ \dot{y} &= -x^3, \end{aligned}$$

but that it is an unstable point of equilibrium for the linearised system there. (Hint: Consider Lyapunov functions of the form $Q = x^m + cy^n$.)