3. Local analysis

$$\dot{x} = f(x), \quad x_0 \text{ s.t } f(x_0) = 0 \qquad (1)$$

Problem: stability of x_0 ?

Perturbation:

$$\dot{x} = f(x) \qquad x = x_0 + \xi$$

$$\Rightarrow \dot{\xi} = f(x_0+\xi) = Df(x_0)\,\xi + \underset{\text{h.o.t}}{\underbrace{O(|\xi|^2)}}$$

$$f_i(x_0+\xi) = \underset{0}{\underbrace{f_i(x_0)}} + \sum_j \left.\frac{\partial f_i}{\partial x_j}\right|_{x=x_0} \xi_j + \text{h.o.t}$$

$Df(x_0)$: Jacobian matrix of f at x_0.

$$\dot{\xi} = Df(x_0)\,\xi + O(|\xi|^2) \qquad (1^*)$$

$$\dot{\xi} = Df(x_0)\,\xi \qquad (2)$$

(2) is the variational eq.

Since x_0 is cst $\Rightarrow Df(x_0)$ is a constant matrix.

We know from Chap. 1 the stability of $\xi = 0$ for (2) is given by Spec$(Df(x_0))$

Q: Stability of x_0 for $\dot{x} = f(x_0)$

$$\overset{?}{\Longleftrightarrow}$$

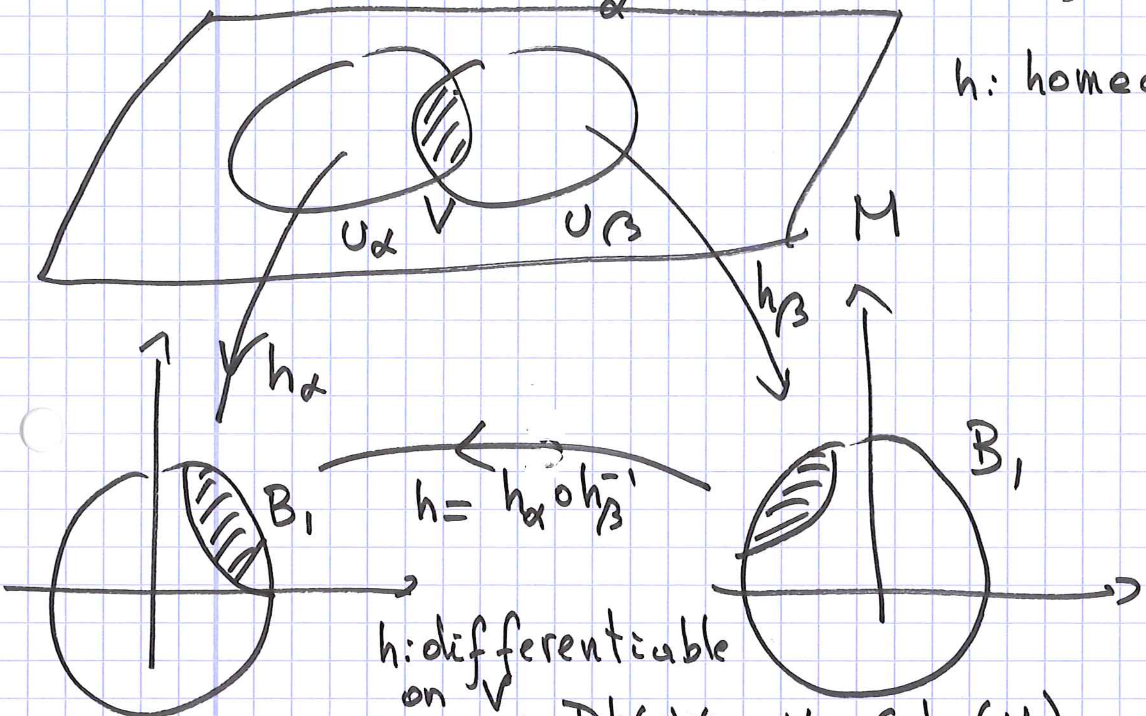
Stability of $\xi = 0$ for $\dot{\xi} = Df(x_0)\,\xi$

3.1 Stable manifold theorem

Differentiable manifold $\mathcal{M}$

$$\mathcal{M} = \bigcup_{\alpha} U_{\alpha} \quad \text{covering}.$$

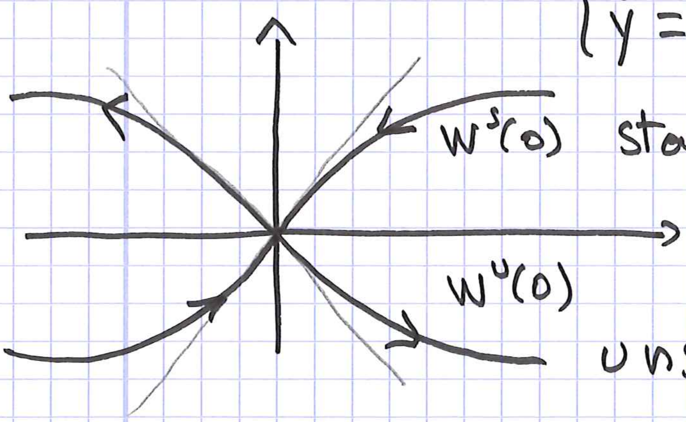
h: homeomorphism.



h: differentiable on V

$\Rightarrow Dh(x) \neq 0 \;\; \forall x \in h_\beta(V)$

Idea in $\mathbb{R}^2$:
$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases}$$
WLOG Let $x=y=0$ be a fixed pt.



$W^s(0)$ stable m'fld of 0

$W^u(0)$ unstable m'fld of 0.

$$W^s(0) = \{ (x,y) \in \mathbb{R}^2 \mid \varphi_t(x,y) \underset{t\to\infty}{\longrightarrow} (0,0) \}$$

$$W^u(0) = \{ (x,y) \in \mathbb{R}^2 \mid \varphi_t(x,y) \underset{t\to-\infty}{\longrightarrow} (0,0) \}$$

How are these m'flds related to E^u, E^s of the system $\dot{\xi} = D(\underline{f})\xi$?

Thm (stable manifold)

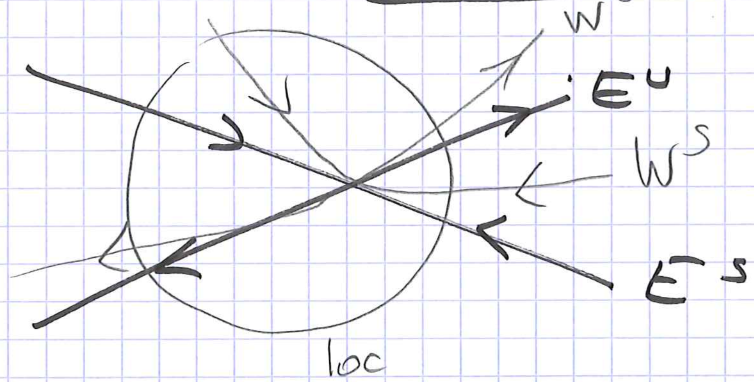
Let $\varphi_t : E \to E$ be the flow of $\dot{x} = f(x)$ w/ fixed pt x_0. Suppose that the spectrum of $Df(x_0)$ is composed of k eigenvalues w/ positive real parts and $(n-k)$ ——— — negative ———.

Then, $\exists$ (in a nbhd of x_0) a k-dim m'fld $W^u_{loc}(x_0)$ tangent to E^u
$(n-k)$ ——— $W^s_{loc}(x_0)$ ——— E^s

s.t. $\forall x \in W^s_{loc}$, $t \geq 0$, $\varphi_t(x) \xrightarrow[t \to \infty]{} x_0$

W^u_{loc} $t \leq 0$ ——— $t \to -\infty$

Moreover, W^u_{loc}, W^s_{loc} are as smooth as f.



Local $\to$ global.

Since $W^{s,u}_{loc}$ exist we define

$$W^s(x_0) = \bigcup_{t \leq 0} \varphi_t(W^s_{loc}(x_0))$$

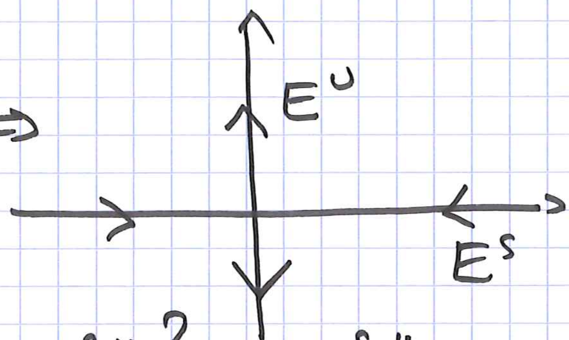
$$W^u(x_0) = \bigcup_{t \geq 0} \varphi_t(W^u_{loc}(x_0))$$

The stable and unstable m'flds.

Example

$$\begin{cases} \dot{x} = -x - y^2 \\ \dot{y} = y + x^2 \end{cases} \qquad x = y = 0 \text{ fixed pt.} \qquad D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow$



$W^{s,u}$? $\quad W^{s,u}_{loc}$ tgt to $E^{s,u}$ so

$$y = \sum_i a_i x^i \quad \text{w/ } a_0 = a_1 = 0$$

$$y = a_2 x^2 + O(x^3)$$

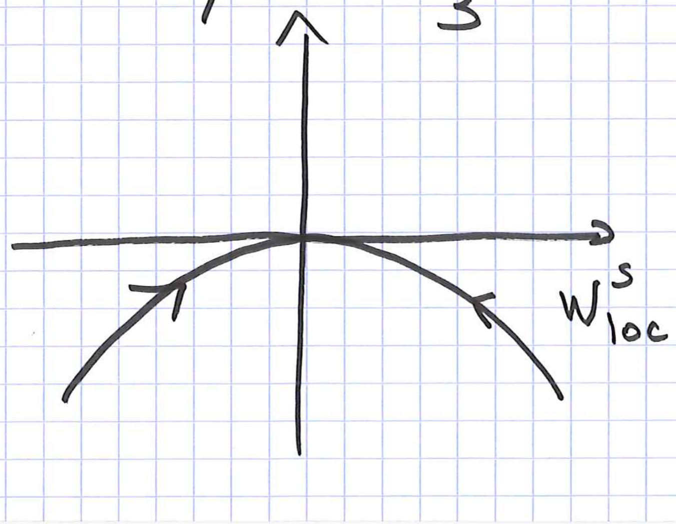
NB $W^{s,u}_{loc}$ analytic since f analytic.

a_2? $\quad \dot{y} = 2a_2 x \dot{x} + O(\dot{x} x^2)$

$$\begin{array}{ccc} \dot{y} & = & 2a_2 x \dot{x} \\ \| & & \| \\ y + x^2 & = & 2a_2 x(-x - y^2) \\ \| & & \| \\ a_2 x^2 + x^2 & = & 2a_2 x(-x - \cancel{a_2^2 x^4}) + O(x^3) \end{array}$$

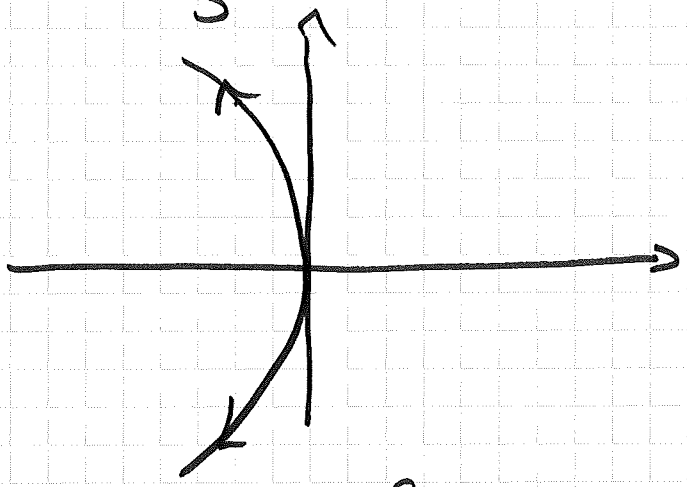
$\Rightarrow \quad a_2 = -1/3$

$$\Rightarrow \quad y = -\frac{x^2}{3} + O(x^5)$$



Same for W^u_{loc}

$$x = -\frac{y^2}{3} + O(y^5)$$

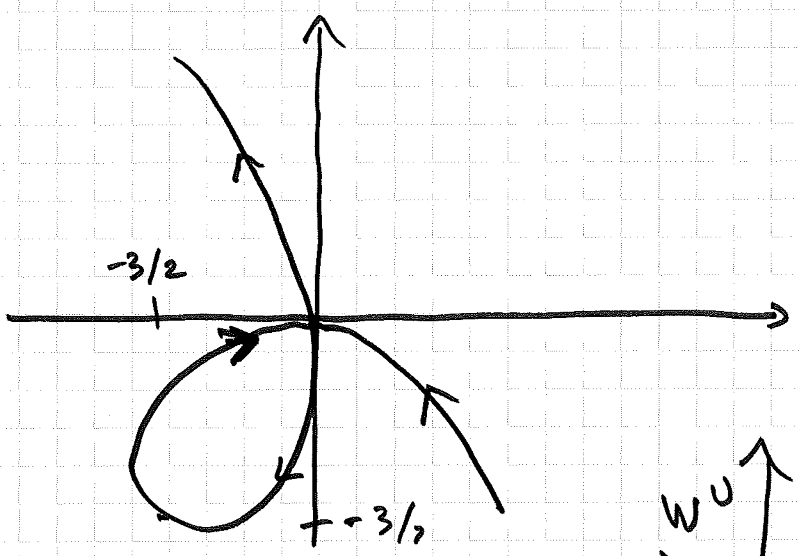


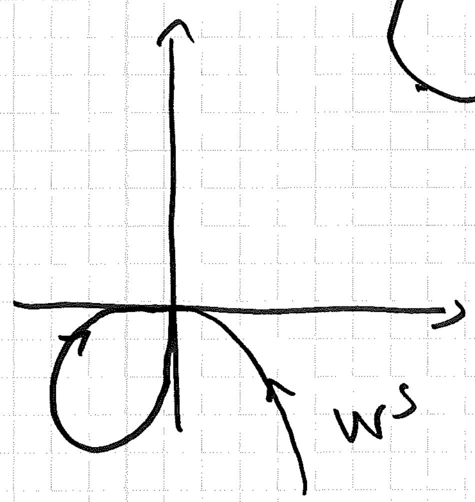
Global manifold?

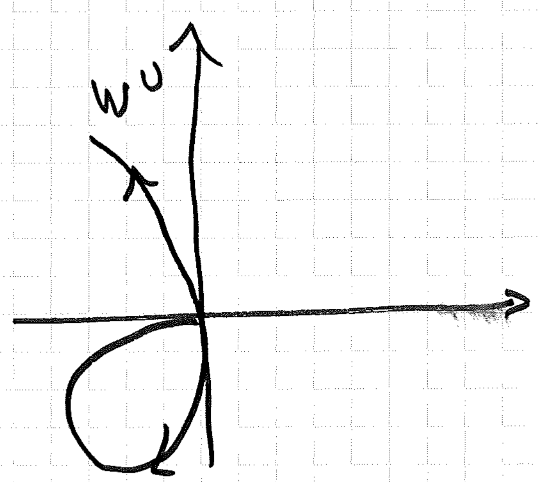
Note $3xy + y^3 + x^3 = 0$ is conserved.

$$J = 3xy + y^3 + x^3 \Rightarrow \dot{J} = 0 \text{ if } J = 0$$

Mclaurin trisectrix (1742)







W^s, W^u not soln curves!

$W^s \cap W^u$: homoclinic m'fld if $\neq \emptyset$.

x_0 is a <u>hyperbolic fixed pt</u> if

$$\mathrm{Re}(\lambda) \neq 0 \quad \forall \lambda \in \mathrm{Spec}(Df(x_0))$$

<u>Thm</u>: x_0 is as. stable if $\mathrm{Re}(\lambda) < 0 \; \forall \lambda$.

<u>Remark</u>: f analytic $\Rightarrow W^{u,s}$ analytic

$\not\Rightarrow$ soln curves are analytic.

$$\dot{x} = f(x) \longrightarrow \dot{\xi} = Df(x_0)\,\xi$$

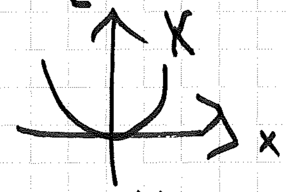
$$? \; \xi = h(x)$$

The stable mfld thm does not guarantee that there is a local change of variables s.t. the system can be linearized locally.

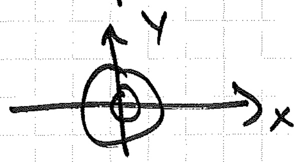
Necessity of hyperbolicity:

$$\ddot{x} + \epsilon x^2 \dot{x} + x = 0$$ damped harmonic oscillator.

$$\Leftrightarrow \begin{cases} \dot{x} = y \\ \dot{y} = -x - \epsilon x^2 y \end{cases} \Rightarrow Df(0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$



$\begin{cases} \text{linear system stable. } \ddot{x} = -x \\ \epsilon = 0 \end{cases}$



$\epsilon \neq 0$? no information from the linearization.

Lyapunov:

$$\mathcal{L} = \frac{y^2}{2} + \frac{x^2}{2} + \alpha x y$$

$$\Rightarrow \dot{\mathcal{L}} = -y^2 \underbrace{\left(\frac{\epsilon x^2}{2} - \alpha\right)}_{>0 \text{ if } \alpha \text{ small}} - \alpha x^2 \underbrace{(1 - \epsilon x y)}_{>0 \ \epsilon \text{ small},\ xy \text{ close to } 0}$$

< 0

$\Rightarrow$ $(0,0)$ is as. stable.

3.2 The center manifold

- $W^{s,u}$ defined as the set of pts s.t $x \underset{t \to \pm\infty}{\longrightarrow} x_0$
- What happens if $\mathrm{Re}(\lambda) = 0$?

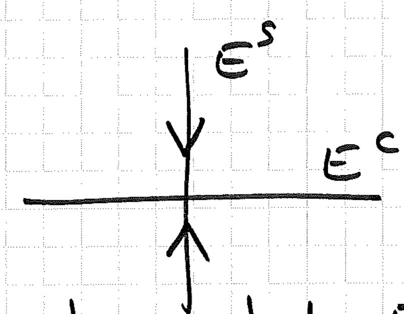
Naive idea: Define W^c as the orbits tangent to E^c

<u>Problem</u>: This set is <u>not</u> unique.

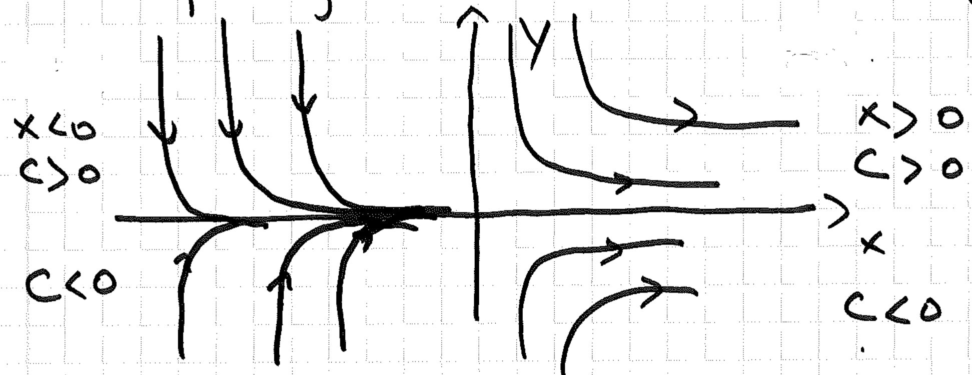
Consider the example:

$$\begin{cases} \dot{x} = x^2 \\ \dot{y} = -y \end{cases} \Rightarrow y = Ce^{1/x}$$

linearized system $\begin{cases} \dot{x} = 0 \\ \dot{y} = -y \end{cases}$



Every single curve $Ce^{1/x}$ is tangent to E^c



Indeed $y' = -\frac{1}{x^2} Ce^{-1/x} \underset{x \to 0^-}{\longrightarrow} 0$

<u>But</u> only one curve as the same smoothness as the vector field $(C = 0)$

Thm. Let $f \in C^r(E)$, $E \subset \mathbb{R}^n$, $r \geq 1$, $f(x_0) = 0$

Assume $Re(\lambda_i) = 0$, $i = 1, \dots, k$,

$\lambda_i \in Spec\{Df(x_0)\}$

$\Rightarrow$ $\exists$ a unique k-dimensional local manifold $W^c_{loc}(x_0)$ tangent to E^c at x_0 and of class C^r and invariant under the flow.

NB. The difference between 2 centre m'fld is transcendental (terms going to 0 faster than any power of x)

In general $f \in C^r(E)$, $E \subset \mathbb{R}^n$, $r \geq 1$

$f(x_0) = 0$ $Spec\{Df(x_0)\} = \Lambda$

k_u eigenvalues s.t $Re(\lambda) > 0$

k_s ———————— < 0

k_c ———————— $= 0$

with $k_u + k_s + k_c = n$

$\Rightarrow$ $\exists$ a $k_{u,s}$-dim m'fld $W^{u,s}_{loc}$ tangent to $E^{u,s}$ at x_0

$\Rightarrow$ $\exists$ a k_c-dim m'fld W^c_{loc} of Class r tangent to E^c at x_0.

3.3 Reduction to centre manifold

Assume that $W^u = \emptyset$ at x_0

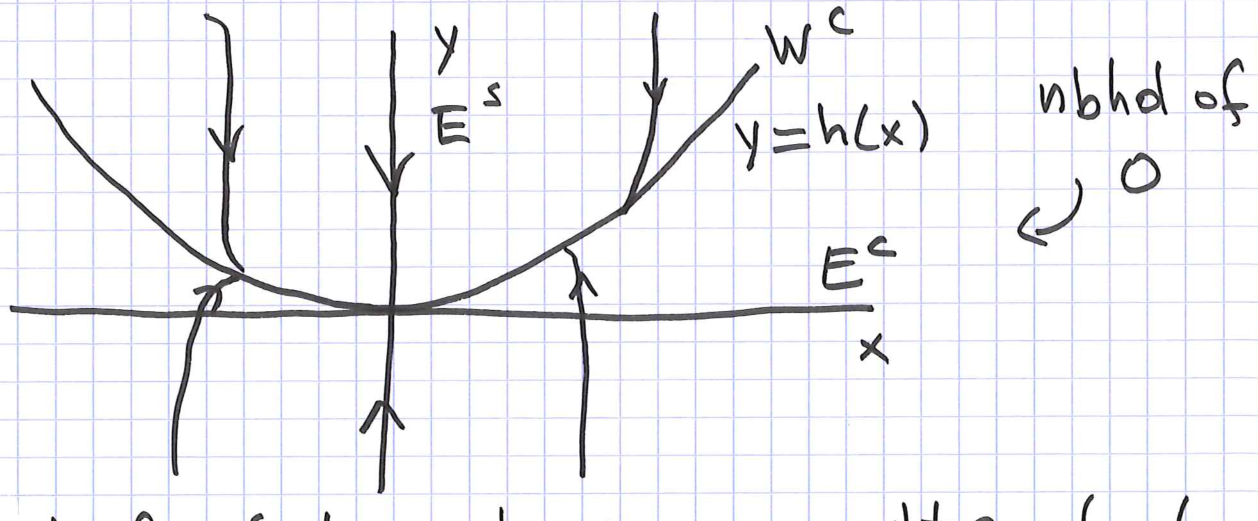
Question: Stability of x_0 in the presence of a centre m'fld ?

Basic idea: Reduction to the centre m'fld.

We start with a system (after linear transf.) of the form:

$$\begin{cases} \dot{x} = Ax + f(x,y) & x \in \mathbb{R}^{\dim W^c} \\ \dot{y} = By + g(x,y) & y \in \mathbb{R}^{\dim W^s} \end{cases}$$

$Re(\lambda) = 0 \quad \forall \lambda \in Spec(A)$
$Re(\lambda) < 0 \quad \forall \lambda \in Spec(B)$



Out of W^c, trajectories are attracted exponentially to W^c

$\Rightarrow$ Dynamics on W^c determines the stability !

$\Rightarrow$ Posit $y = h(x)$ and solve for $h(x)$

$$y = h(x) \Rightarrow \dot{y} = Dh(x)\,\dot{x}$$

$$\Rightarrow \begin{cases} \dot{x} = Ax + f(x, h(x)) \\ \dot{y} = Dh(x)\,\dot{x} = Bh(x) + g(x, h(x)) \end{cases}$$

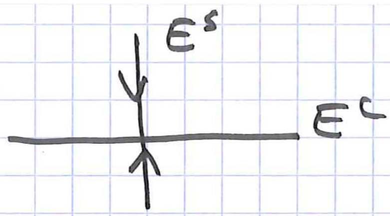
Second eqn:

$$(*) \quad \boxed{Dh(x)\left[Ax + f(x, h(x))\right] = Bh(x) + g(x, h(x))}$$

This is an equation for $h(x)$ that can be solved by expanding $h(x)$ in Taylor series in a nbhd of 0.

For instance, if $\left\{ \begin{matrix} f \\ g \end{matrix} \right.$ is polynomial then truncating Dh an h to a given order gives conditions on the coef h_m

(Here $\quad h(x) = \sum_{|m|=0}^{d} h_m x^m + O(|x|^{d+1})$)



Example:

$$\begin{cases} \dot{x} = x^2 y - x^5 \\ \dot{y} = -y + x^2 \end{cases} \qquad (x,y) \in \mathbb{R}^2$$

$(x,y) = (0,0)$ f. pt. w/ eig. $(0,-1)$

Naive idea: set $y=0 \Rightarrow \dot{x} = -x^5$ stable ?

Center mfld:

$$y = h(x) = h_2 x^2 + h_3 x^3 + O(x^4)$$

$$\Rightarrow Dh(x) = 2\,x + 3 h_3 x^2 + O(x^3)$$

$\Rightarrow$ (*) reads:

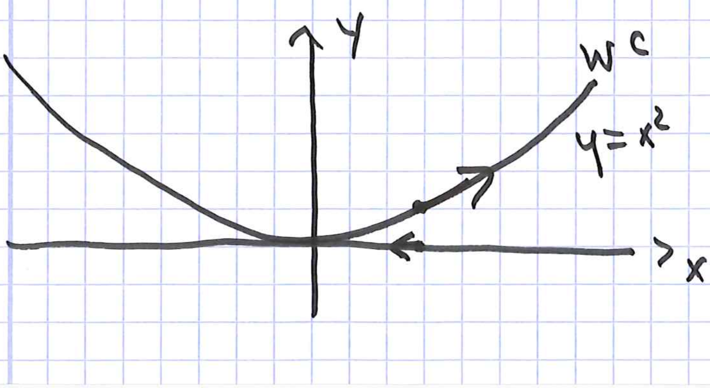
$$(2h_2 x + 3h_3 x^2)(x^2(h_2 x^2 + h_3 x^3) - x^5)$$
$$= -(h_2 x^2 + h_3 x^3 - x^2) + \text{h.o.t}$$

$$\Rightarrow h_2 = 1, \quad h_3 = 0$$

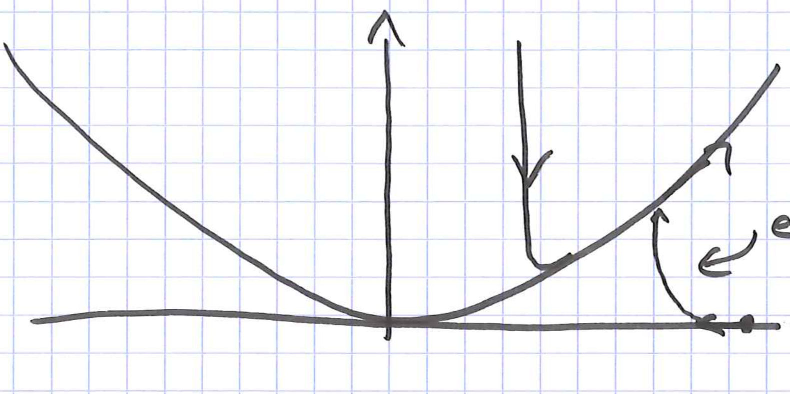
$$\Rightarrow h(x) = x^2 + O(x^4)$$

On the centre m'fld, the dynamics is

$$\dot{x} = x^2(x^2 + O(x^4)) + x^5$$
$$= x^4 + O(x^5) \quad \Rightarrow x = 0 \text{ is } \underline{\text{unstable}}$$



$\Rightarrow (0,0)$ is unstable.



Thm 1: The dynamics on W^c is for (x,y) close enough to 0 given by

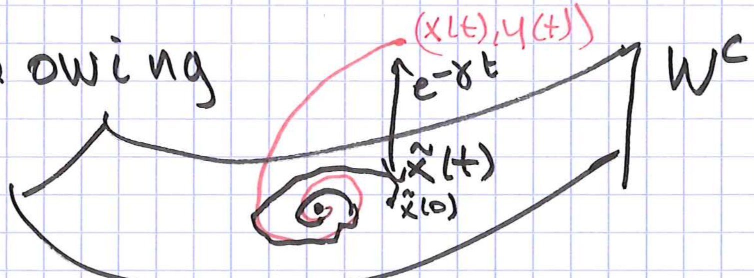
$$\dot{\tilde{x}} = A\tilde{x} + f(\tilde{x}, h(\tilde{x}))$$

Thm 2: Let (x_0, y_0) be small enough. Then $\forall$ $(x(t), y(t))$ based on (x_0, y_0), $\exists$ a soln $\tilde{x}(t)$ s.t.

$$x(t) = \tilde{x}(t) + O(e^{-\gamma t})$$
$$y(t) = h(\tilde{x}(t)) + O(e^{-\gamma t})$$

for some constant $\gamma > 0$

$\Rightarrow$ Shadowing



Close enough to 0, each trajectory has a shadow on W^c and converges to that trajectory exponentially

Thm 3: Let $\phi_q(x)$ be the Taylor approx of the centre mfld of degree q

$$\Longrightarrow |h(x) - \phi_q(x)| = O(|x|^q)$$

as $x \to 0$.

3.4 Mappings

$x \mapsto G(x)$ $\quad x \in E \subset \mathbb{R}^n$ $\quad$ Assume G^{-1} exists

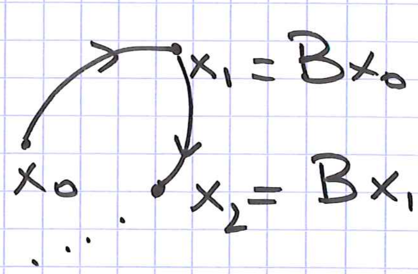
Linear maps:

Another dynamical system:

$$x_{n+1} = B x_n \qquad B \in \mathcal{M}_n(\mathbb{R}), n \in \mathbb{Z}$$

sends pts to pts

If $0 \notin \text{spec}(B)$, orbits are unique



$$\begin{cases} \lambda_j = a_j + i b_j & \lambda_j \in \text{Spec}(B) \\ w_j = u_j + i v_j & u_j, v_j \in \mathbb{R}^n \end{cases}$$

$$\Rightarrow E^s = \{u_j, v_j \mid |\lambda_j| < 1\}$$
$$E^u = \{ \qquad\qquad = 1\}$$
$$E^c = \{ \qquad\qquad > 1\}$$

If $x_0 \in E^s \Rightarrow \exists\, \alpha, C$ s.t $|x_n| \le C\alpha^n |x_0|$ $\quad x_0 \in E^s$

Note $\varphi_t = e^{At}$ define $B = e^{At}$

$$\Rightarrow \varphi_t(x): x_0 \to Bx_0$$

Every linear flow defines a linear map

$\cancel{\Leftarrow}$

Stability of maps Let x_0 s.t. $G(x_0) = x_0$

1/ Stability of x_0: $\forall \epsilon > 0, \exists \delta > 0$ s.t $\forall x \in B_\delta(x_0)$
$G^n(x) \in B_\epsilon(x_0)$ $n \in \mathbb{Z}^+$

2/ Asymptotic stability of x_0 : stability and
$\exists \delta$ s.t $\forall x \in B_\delta : G^n(x) \xrightarrow[n \to \infty]{} x_0$

Stable and unstable m'flds:

Local stable & unstable m'fld

$W^{u,s}_{loc}(x_0)$ are tangent to $E^{u,s}(x_0)$

Again they have the same dimension & smoothness

$$W^s_{loc}(x_0) = \{x \in U \mid G^n(x) \to x_0, n \to \infty; G^n(x) \in U, \forall n \geq 0\}$$
$$W^u_{loc}(x_0) = \{\text{———————} \; n \to -\infty \; \text{————} \leq 0\}$$

By extension:

$$W^s(x_0) = \bigcup_{n \geq 0} G^{-n} W^s_{loc}(x_0)$$

$$W^u(x_0) = \bigcup_{n \geq 0} G^n W^s_{loc}(x_0)$$

⚠ M'flds are unions of trajectories not trajectories themselves.

Ex: A 2D linear map

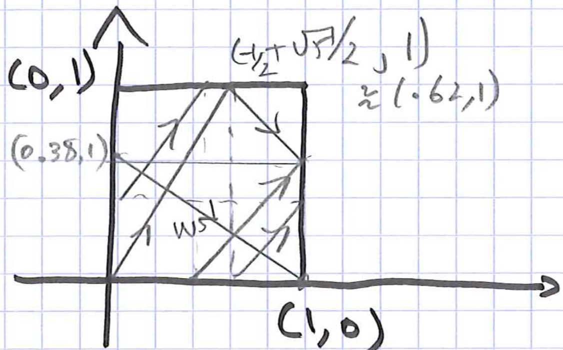
$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}}_{B} \begin{bmatrix} x \\ y \end{bmatrix} \qquad (x,y) \in T^2 = \mathbb{R}^2/\mathbb{Z}^2$$

$$B = \{ [0,1], [0,1] \} \circlearrowleft$$

$$\lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2}$$

$$v_1 = \left[\frac{-1+\sqrt{5}}{2}, 1 \right]^T$$

$$v_2 = \left[\frac{-1-\sqrt{5}}{2}, 1 \right]$$



Stable & unstable m'flds fill densely the torus.

Since B has rational entries, a rational # is sent to a rational #.

Periodic orbits?

$$x_0 = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \to x_1 = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} \to x_2 = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} \to x_3 = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

$\Rightarrow$ Period 3

# of orbits period 2 : 4
3 : 15
4 : 44
5 : 120
6 : 319
7 : 840
⋮

$\Rightarrow$ Infinitely many periodic pts.
The set of such pts is dense in T^2.

Stability of periodic orbits

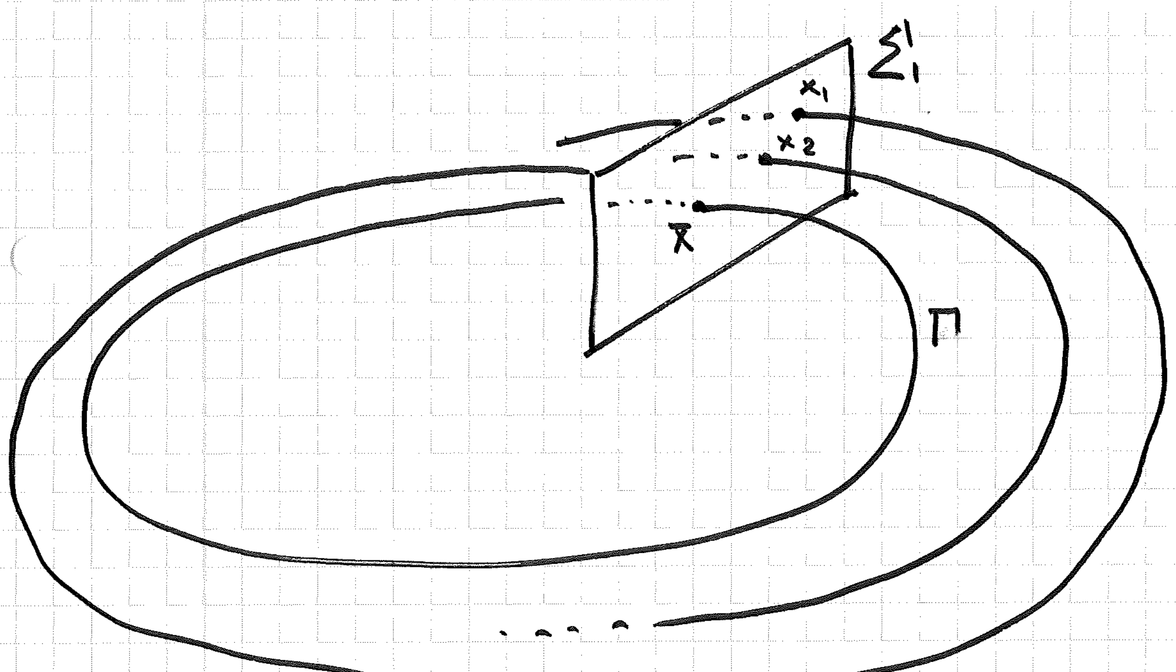
A per. orbit Γ is a closed curve in phase space. It is **stable** if $\forall \epsilon > 0$, $\exists \delta > 0$ & nbhd U_δ of Γ s.t $\forall x \in U_\delta$, $d(\varphi_t(x), \Gamma) < \epsilon$

($d(\varphi_t(x), \Gamma)$ is the distance between $\varphi_t(x)$ and Γ).

Γ is **as. stable** (a limit cycle) if it is stable and $\exists\, \delta > 0$ s.t

$$\lim_{t \to \infty} d(\varphi_t(x), \Gamma) = 0 \quad \forall x \in U_\delta$$

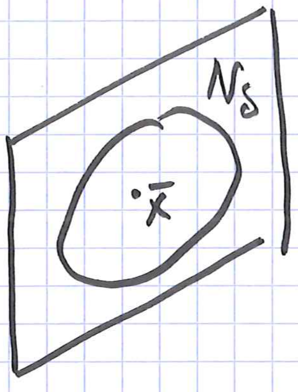
Maps $\Longleftrightarrow$ Periodic orbits



$x \longmapsto P(x)$: Poincaré map

Condition: $\varphi_{t_1}(x_0) \in \Sigma_1$

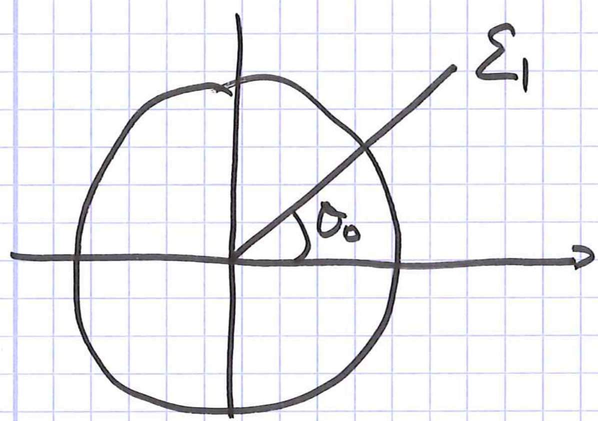
$\vec{n} \cdot \vec{f}(x_1) > 0$ (transversal)



$$\Rightarrow \exists\, \delta > 0 \text{ s.t. } \forall x \in N_\delta(\bar{x})$$
$$\exists\, T(x) \text{ s.t. } \varphi_{T(x)}(x) \in \Sigma_1$$

Ex: $\begin{cases} \dot{r} = r(1-r^2) \\ \dot{\theta} = 0 \end{cases}$

$$\Rightarrow \begin{cases} r(t, r_0) = \left[1 + \left(\frac{1}{r_0^2} - 1\right) e^{-2t}\right]^{-1/2} \\ \theta(t, \theta_0) = t + \theta_0 \end{cases}$$



Let $p_0 = (r_0, \theta_0) \in \Sigma_1$

$\Rightarrow \varphi_{2\pi}(p_0) \in \Sigma_1$

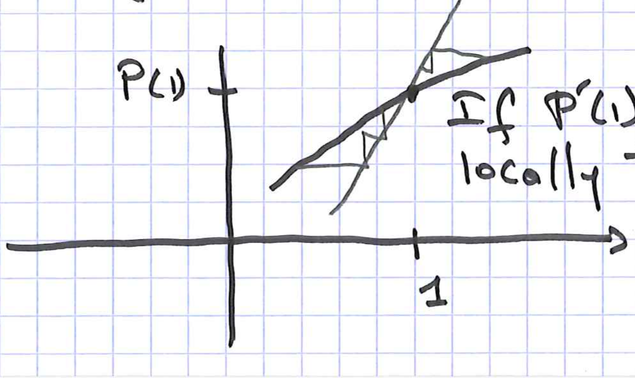
$\Leftrightarrow T(p) = 2\pi \quad \forall p \in \Sigma_1$

$$P(r) = \left[1 + \left(\frac{1}{r^2} - 1\right) e^{-4\pi}\right]^{-1/2}$$

$P(1) = 1$ (fixed pt)

Stability? $P'(1) = e^{-4\pi} < 1 \Rightarrow$ Stable

Geometrically:



If $P'(1) < 1$
locally traj
accumulate