B5.6: Nonlinear Systems-Sheet 4 (solutions)

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Q1 (limit cycles) We consider the system given by

$$\begin{aligned} \dot{x} &= -y + xf(\sqrt{x^2+y^2}), \\ \dot{y} &= x + yf(\sqrt{x^2+y^2}), \end{aligned} \tag{1}$$

where $f(r) = \sin r$. By denoting $r^2 = x^2 + y^2$ the corresponding dynamical system in polar coordinates $r(t)$ and $\theta(t)$ takes the form:

$$r\dot{r} = x\dot{x} + y\dot{y} \implies \dot{r} = r\sin(r), \tag{2}$$

and

$$\begin{aligned} \dot{\theta} &= \frac{d}{dt}\left(\tan^{-1}\left(\frac{y}{x}\right)\right) \\ \dot{\theta} &= \frac{x\dot{y} - \dot{x}y}{x^2+y^2} \implies \dot{\theta} = 1, \end{aligned}$$

which suggests that the trajectories rotate anti-clockwise around the origin.

Determination of the fixed points and periodic orbits, along with their stability, is found from (2). In particular, the fixed point $(0, 0)$ is an unstable spiral whilst there are asymptotically stable periodic orbits at radii $r = (2n-1)\pi$ and unstable periodic orbits for $r = 2\pi n$ for $n \in \mathbb{Z}^+$. If we were to consider negative radii, the opposite results would apply, given that (2) is an even function (see Fig. 1); explicitly, asymptotically stable periodic orbits occur at $r = 2n\pi$ and unstable periodic orbits at $r = (2n-1)\pi$ for $n \in \mathbb{Z}^-$. Phase-plane dynamics of (1) for positive and negative radii is plotted in Fig. 1.

Q2 (a bead on a wire) Consider the equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta - w^2\sin\theta\cos\theta = 0, \tag{3}$$

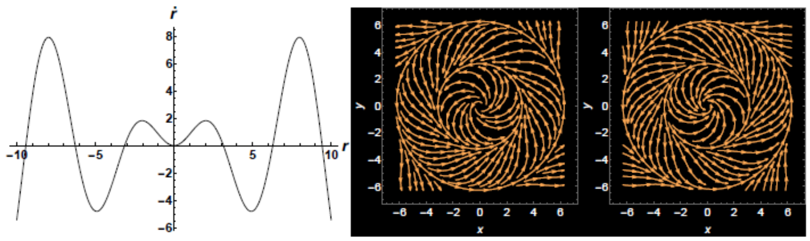

Figure 1: Plot of (2) and phase planes of (1) for $r > 0$ (shown left) and $r < 0$ (shown right).

which can be further simplified by rescaling time $t\sqrt{g/L} \to t$ for g, $L > 0$, and regrouping the parameters:

$$\frac{d^2\theta}{dt^2} + \sin\theta - 4\alpha\sin\theta\cos\theta = 0,$$

where $\alpha = \omega^2 L/4g$.

(i) : We separate it into the dynamical system:

$$\dot{\theta} = y, \dot{y} = -\sin\theta + 4\alpha\sin\theta\cos\theta, \tag{4}$$

which has fixed points given by $y = 0$ and:

$$\begin{aligned} \sin\theta &= 0, \\ \cos\theta &= \frac{1}{4\alpha}, \end{aligned} \tag{5}$$

the latter of which is unsolvable unless $\alpha \geq 1/4$.

The behavior of the system is as follows: At $\alpha = \omega = 0$, there are three fixed points $\theta = \{-\pi, 0, \pi\}$, which are unstable (saddle by linearization), Lyapunov stable (circle by linearization), and unstable (saddle by linearization) respectively. As α increases, there is a supercritical pitchfork bifurcation at $\alpha = 1/4$. The Lyapunov stable solution of $\theta = 0$ now becomes unstable (saddle node) and two Lyapunov stable solutions (centers), obtained by solving (5), are introduced. Both of these branches tend to $\pm\pi/2$ as $\alpha \to \infty$ (see Fig. 2).

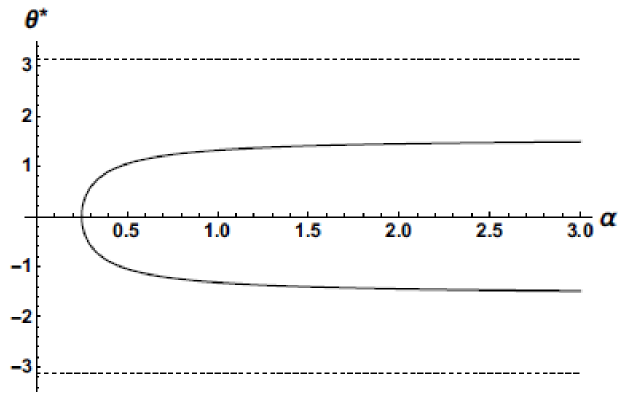

Figure 2: Bifurcation diagram of (3) with solid and dashed lines corresponding to Lyapunov stable and unstable solutions, respectively.

(ii) : We can rewrite (3) as

$$\frac{d^2\theta}{dt^2} + \sin\theta - 2\alpha\sin(2\theta) = 0,$$

which has the first integral

$$\frac{1}{2}\dot{\theta}^2 + V(\theta) = E, \tag{6}$$

for integration constant E and potential $V(\theta) = -\cos\theta + \alpha\cos(2\theta)$.

For the bead to continually circle the hoop in one direction, we require more energy than the maximum value of $V(\theta)$ which is $\alpha + 1$. Trajectories that initially start at $\theta = \pi/2$ correspond to $V = -\alpha$. Therefore, from (6):

$$\frac{1}{2}\dot{\theta}^2 + V\left(\frac{\pi}{2}\right) = \alpha + 1 \implies \dot{\theta} = \sqrt{2(2\alpha+1)},$$

is the minimum velocity of the bead.

As such, if the bead is initially at $\theta = \pi/2$, then the bead will continually circle the hoop provided $|\dot{\theta}| > \sqrt{2(2\alpha+1)}$. Note that the equality corresponds to the bead tending to the potential maximum as $t \to \infty$.

(iii) : Adding linear dampening to (3) we obtain:

$$\frac{d^2\theta}{dt^2} - \mu\frac{d\theta}{dt} + \frac{g}{L}\sin\theta - w^2\sin\theta\cos\theta = 0,$$

with dampening corresponding to $\mu < 0$. We rescale $L\mu/g \to \mu$ and $t\sqrt{g/L} \to t$ to obtain from (7):

$$\frac{d^2\theta}{dt^2} - \mu\frac{d\theta}{dt} + \sin\theta - 4\alpha\sin\theta\cos\theta = 0, \tag{7}$$

where $\alpha = \omega^2 Lg/4$, which can be split into the dynamical system:

$$\dot{\theta} = y, \dot{y} = \mu y - \sin\theta + 4\alpha\sin\theta\cos\theta, \tag{8}$$

which has the same fixed points given by (5). Taking the Jacobian of the latter system, we find:

$$Df(\theta, y) = \begin{bmatrix} 0 & 1 \\ -\cos\theta + 4\alpha\cos 2\theta & \mu \end{bmatrix}. \tag{9}$$

We note from (9) that, for dampening $\mu < 0$ case $\alpha < 1/4$ now corresponds to a stable node whilst $\alpha > 1/4$ gives a saddle; that is, $\alpha = 1/4$ is the critical value for the appearance of an instability at this fixed point. Phase plot portraits of (7) with $\mu = -1$ for $\alpha = 0.1$ and $\alpha = 1$ are shown in Fig. 3.

Q3 (centre manifold to the third order) Consider the system

$$\dot{x} = y - x - x^2, \tag{10}$$
$$\dot{y} = \mu x - y - y^2. \tag{11}$$

The Jacobian at origin is given by

$$Df(0,\, 0) = \begin{bmatrix} -1 & 1 \\ \mu & -1 \end{bmatrix}.$$

and has eigenvalues given by:

$$\lambda = -1 \pm \sqrt{\mu},$$

which suggests that a bifurcation occurs at the origin when $\mu = 1$, for this is a particular value associated with $\mathrm{Re}(\lambda) = 0$.

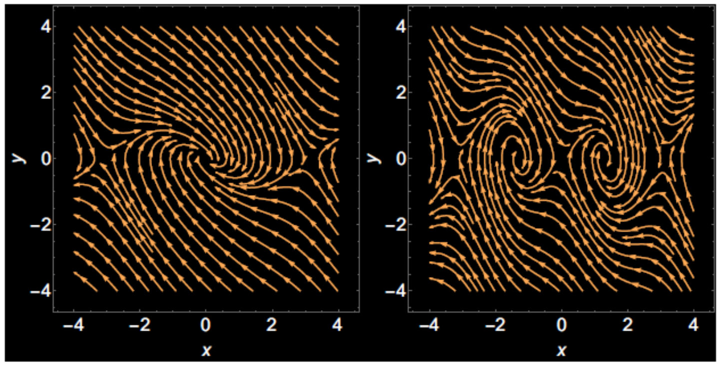

Figure 3: Phase-plane plots of (7) for $\mu = -1$ and $\alpha = 0.1$ (shown left) and $\alpha = 1$ (shown right).

The associated linear subspaces are:

$$E^s = \text{span}\{(-1,\, 1)^T\},\, E^c = \text{span}\{(1,\, 1)^T\}.$$

We suppose the center manifold at the origin can be expressed to third order as:

$$y = h(x) = x + a_2x^2 + a_3x^3 + O(x^4), \tag{12}$$

where $a_0 = 0$ and $a_1 = 1$, so that the extended center manifold intersects with E^c at the origin and lies tangent to it. Substituting (12) into (11), we find:

$$(a_2-1)x^2+(a_3+2a_2^2-2a_2)x^3+O(x^4) = (-1-a_2)x^2+(-2a_2-a_3)x^3+O(x^4),$$

which suggests that $a_2 = 0$ and $a_3 = 0$, so that the center manifold at $\mu = 1$ is given by:

$$y = x + O(x^4). \tag{13}$$

Substituting (13) into (10) gives the reduced evolution on the center manifold:

$$\dot{x} = -x^2, \tag{14}$$

The phase-plane plot of (10)-(11) for $\mu = 1$ is shown in Fig. 4. Additionally, analysing the extended center manifold for $\mu \neq 0$ one can show that system (10)-(11) undergoes a transcritical bifurcation at the origin for $\mu = 1$.

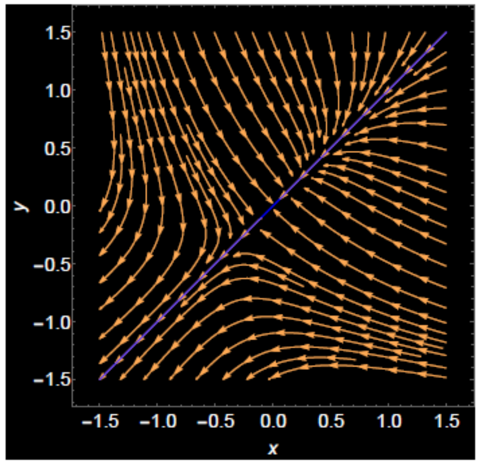

Figure 4: Phase-plane plot of (10)-(11) for $\mu = 1$ along with the center manifold (14) (shown in blue).

Q4 (a 1D map) We consider the map

$$x_{n+1} = (1+\mu)x_n - \mu x_n^2 = f(x_n, \mu), \tag{15}$$

for $\mu \geq 0$.

(i) : To find the fixed points x^*, we solve the equation:

$$x^* = (1+\mu)x^* - \mu(x^*)^2,$$

which gives the solutions $x^* = \{0,\, 1\}$.

We calculate the corresponding stability by determining $\partial f/\partial x$ evaluated at x^*. From (15), we find:

$$\begin{aligned} \frac{\partial f}{\partial x}(x^*) &= (1+\mu) - 2\mu x^*, \\ \frac{\partial f}{\partial x}(0) &= 1+\mu, \quad \frac{\partial f}{\partial x}(1) = 1-\mu, \end{aligned} \tag{16}$$

which implies that $x^* = 0$ is unstable for all $\mu > 0$ and $x^* = 1$ is stable for $\mu \in (0, 2)$ and unstable for $\mu > 2$.

(ii) : To determine the period-2 cycles, we solve the equation:

$$\begin{aligned} x^* &= f(f(x_n;\mu)) = (1+\mu)\left[(1+\mu)x^* - \mu(x^*)^2\right] - \mu\left[(1+\mu)x^* - \mu(x^*)^2\right]^2, \\ 0 &= \mu x^*\left[(2+\mu) - (2+3\mu+\mu^2)x^* + 2\mu(1+\mu)(x^*)^2 - \mu^2(x^*)^3\right]. \end{aligned} \tag{17}$$

We note that both $x^* = 0$ and $x^* = 1$ are solutions, because any fixed point is also a period-2 cycle. Therefore, we can reduce (17) to

$$\mu^2(x^*)^2 - (\mu^2 + 2\mu)x^* + \mu + 2 = 0,$$

which by solving the quadratic polynomial, implies that the period-2 cycle is comprised of the points:

$$x_{1,2} = \frac{\mu + 2 \pm \sqrt{\mu^2 - 4}}{2\mu}. \tag{18}$$

To determine the stability of this periodic orbit, we calculate:

$$\lambda = \frac{\partial f}{\partial x}(x_2)\frac{\partial f}{\partial x}(x_1). \tag{19}$$

Expanding (19) we find:

$$\begin{aligned} \lambda &= [(1+\mu) - 2\mu x_2][(1+\mu) - 2\mu x_1], \\ &= (1+\mu)^2 - 2\mu(1+\mu)[x_1 + x_2] + 4\mu^2 x_1 x_2, \\ &= 5 - \mu^2. \end{aligned} \tag{20}$$

Therefore, the period-2 cycle is stable if $|5 - \mu^2| < 1$, which suggests that $\mu \in (2, \sqrt{6})$, and unstable for $\mu \in (\sqrt{6}, \infty)$. We show the asymptotic convergence to this periodic orbit on a cobweb plot in Fig. 5.

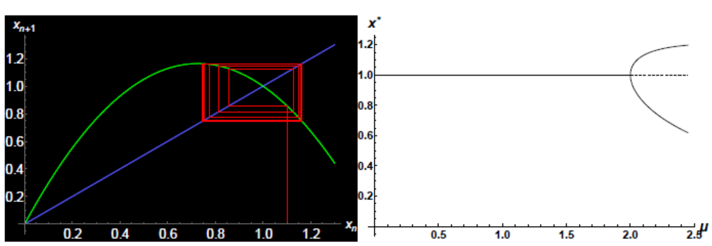

Figure 5: Left: Cob-web plot showing the asymptotic convergence to the period-2 cycle determined by (18) for $x_0 = 1.1$ and $\mu = 2.2$. Right: Analytic bifurcation diagram obtained by using the results (16), (18) and (20).

(iii) and (iv) : Using the results from (i) and (ii), in particular, (16), (18) and (20), we can construct the bifurcation diagram shown in Fig. 5. Note that at $\mu = \sqrt{6}$, stable period-4 cycles branch from the period-2 cycles, which now become unstable. As μ further increases, the period doubling cascades into chaos. (see Fig. 6 for the numerically generated bifurcation diagram).

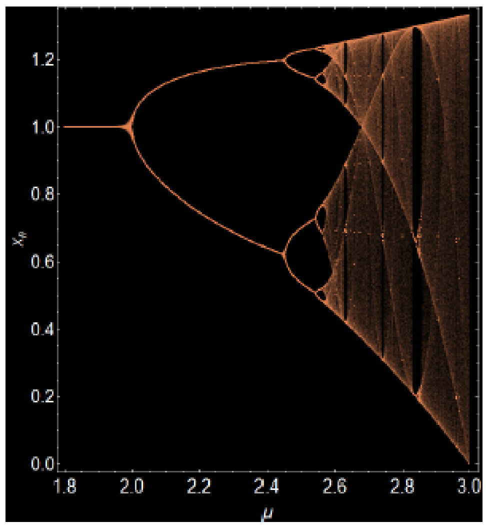

Figure 6: Numerical bifurcation diagram showing the stable orbits of (15).

Q5 (stability of periodic orbits) We consider a 1D map

$$x_{n+1} = f(x_n),$$

and assume that it supports a p-periodic orbit $\{x_1, x_2, ..., x_p\}$ such that $x_i \neq x_j, \quad \forall i, j \in \{1, ..., p\}$ with $i \neq j$, and $x_{p+1} = x_1$.

By the definition of a p-periodic orbit:

$$x_1 = f^p(x_1) = f(x_p).$$

To determine the stability of this periodic cycle, we find:

$$\begin{aligned}
\lambda &= \frac{d}{dx_1}(f^p(x_1)) = \frac{d}{dx_1}(f(x_p)), \\
\text{(Chain rule)} &= \frac{df(x_p)}{dx_p}\frac{dx_p}{dx_1}, \\
\text{(Definition of map)} &= \frac{df(x_p)}{dx_p}\frac{df(x_{p-1})}{dx_1}, \\
\text{(Repeating)} &= \frac{df(x_p)}{dx_p}\frac{df(x_{p-1})}{dx_{p-1}}...\frac{df(x_1)}{dx_1},
\end{aligned}$$

which gives the required result that

$$\lambda = \prod_{i=1}^{i=p} f'(x_i)$$

determines the stability.