B5.6: Nonlinear Systems-Sheet 5 (solutions)

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Q1 (center manifold to fourth order) We consider the system

$$\dot{x} = xy + ax^3 + xy^2, \tag{1}$$
$$\dot{y} = -y + bx^2 + x^2y, \tag{2}$$

(i) : The linearisation at origin gives:

$$Df(0, 0) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \tag{3}$$

and is independent of values of parameters $a, b \in \mathbb{R}$. Therefore, we construct usual center manifold at origin for fixed values of a, b in the form:

$$y = h(x) = c_2x^2 + c_3x^3 + c_4x^4 + O(x^5),$$

where $c_0 = c_1 = 0$ so that the center manifold intersects the origin and lies tangent to the center subspace spanned by $(1, 0)^T$. We substitute $y = h(x)$ into (1) to obtain:

$$2c_2(c_2 + a)x^4 + O(x^5) = (b - c_2)x^2 - c_3x^3 + (c2 - c4)x^4 + O(x^5),$$

which implies that $c_2 = b$, $c_3 = 0$ and $c_4 = b - 2b(a + b)$ and leads to the center manifold having the form:

$$y = bx^2 + [b - 2b(a + b)]x^4 + O(x^5). \tag{4}$$

We find the reduced equation by substituting (4) into (1):

$$\dot{x} = (a + b)x^3 + (b^2 + b - 2b(a + b))x^5 + O(x^6). \tag{5}$$

We note that the origin is asymptotically stable for $a + b < 0$ and unstable for $a + b > 0$, as required.

(ii) : If $a + b = 0$, then the stability is determined by going to order $O(x^5)$ in (5). In this case, if $b^2 + b > 0$, which suggests that $b > 0$ or $b < -1$, then the origin is unstable, whilst if $b^2 + b < 0$, which suggests that $b \in (-1, 0)$, then the origin is asymptotically stable. However, if $b^2 + b = 0$, such that $b = \{-1, 0\}$, then we need to go to higher order in x to determine the stability. The phase portraits of system (1)-(2) are shown in the case $a + b = 0$ in Fig. 1.

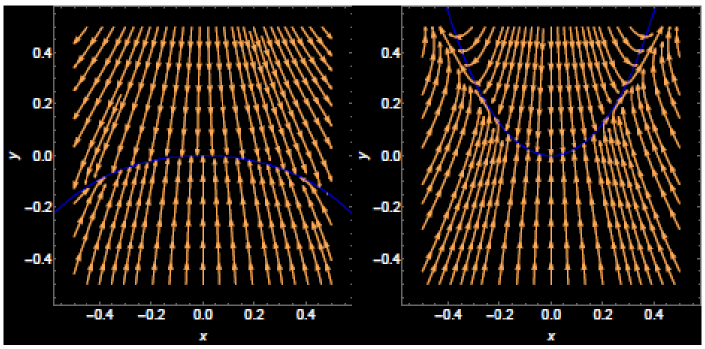

Figure 1: Phase portraits of (1)-(2) for (a, b)=(0.5,-0.5) and (-3, 3) together with the center manifold (4) (shown in blue).

Q2 (topological equivalence) Consider systems

$$\dot{x}_1 = -x_1, \quad \dot{x}_2 = -x_2, \tag{6}$$

and

$$\dot{x}_1 = -x_1, \quad \dot{x}_2 = -2x_2. \tag{7}$$

The solution of (6) is given by

$$x_1(t) = x_1(0)e^{-t}, \; x_2(t) = x_2(0)e^{-t},$$

while one of (7) is

$$\tilde{x}_1(t) = \tilde{x}_1(0)e^{-t}, \; \tilde{x}_2(t) = \tilde{x}_2(0)e^{-2t},$$

The homeomorphism $h(x_1,\, x_2)$ between solutions of (6) and (7) is given by

$$\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = h(\tilde{x}_1,\, \tilde{x}_2) = \begin{bmatrix} x_1 \\ x_2^2 \end{bmatrix}.$$

Therefore, systems (6) and (7) are topologically equivalent, but not smoothly equivalent, because $h^{-1}(\tilde{x}_1,\, \tilde{x}_2)$ involves a square root operation, which is not differentiable at origin. Systems (6) and (7) are also not orbitally equivalent, because there is no time rescaling $\tilde{t} = \mu t$ which transforms system (6) into (7).

Q3 (centre manifold to the third order) Consider the system

$$\begin{aligned} \dot{x} &= \mu x + y + \sin x, && (8) \\ \dot{y} &= x - y. && (9) \end{aligned}$$

The fixed points of it are solutions of system:

$$\begin{aligned} y &= x, \\ -(\mu+1)x &= \sin x, && (10) \end{aligned}$$

implying that origin is a fixed point for all values of $\mu \in \mathbb{R}$ and that a bifurcation occurs at $\mu = -2$, creating new fixed points which satisfy (10). Additionally, (10) predicts that countably infinite fixed points bifurcate at $\mu = -1$ and $x = n\pi$ for $n \in \mathrm{Z}$.

The Jacobian at origin for $\mu = -2$ is given by

$$Df(0,\, 0) = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

and has eigenvalues given by:

$$\lambda_1 = -2,\, \lambda_2 = 0,$$

and eigenvectors

$$v_1 = (-1,\, 1)^T,\, v_2 = (1,\, 1)^T.$$

The associated linear subspaces are:

$$E^s = \mathrm{span}\{(-1,\, 1)^T\},\, E^c = \mathrm{span}\{(1,\, 1)^T\}.$$

To determine the type of bifurcation occurring for $\mu = -2$ it is enough in this problem to calculate the center manifold at the origin for this parameter value:

$$y = h(x) = x + a_2 x^2 + a_3 x^3 + O(x^4), \qquad (11)$$

where $a_0 = 0$ and $a_1 = 1$, so that the center manifold intersects with E^c at the origin and lies tangent to it. Substitution of (11) into system (8)-(9) combined with Taylor expansion of $\sin x$ at the origin results into:

$$-a_2x^2 - a_3x^3 + O(x^4) = a_2x^2 + (a_3 - \frac{1}{6} + 2a_2)x^3 + O(x^4),$$

which suggests that $a_2 = 0$ and $a_3 = 1/12$, so that the center manifold at $\mu = -2$ is given by:

$$y = x + \frac{1}{12}x^3 + O(x^4). \tag{12}$$

Substituting (12) into (8) gives the reduced evolution on the center manifold:

$$\dot{x} = -\frac{1}{12}x^3, \tag{13}$$

suggesting that one has a supercritical pitchfork bifurcation at $\mu = 2$. The phase-plane portraits of (8)-(9) before the bifurcation, at the bifurcation, and after the bifurcation are shown in Fig. 2. To obtain them one needs to calculate the extended center manifolds.

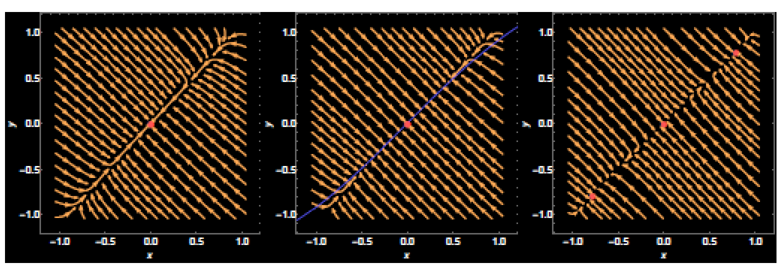

Figure 2: Phase portraits of (8)-(9) for left: $\mu = -2.1$, middle: $\mu = -2.0$, right: $\mu = -1.9$. Fixed points are shown as red dots and the extended center manifolds are shown in blue.

Q4 (pitchfork bifurcation) We consider the equation $\dot{x} = f(x, \mu)$ where $x \in \mathbb{R}$ and f is at least class $C^3(\mathbb{R})$. For a pitchfork bifurcation to occur at $x = \mu = 0$, we demand that:

$$f(0,0) = 0, \tag{14}$$
$$f_x(0,0) = 0, \tag{15}$$
$$f_\mu(0,0) = 0, \tag{16}$$

whereby the latter-most condition is a necessary condition that there is another curve crossing $(x,\, \mu) = (0,\, 0)$. As such, we can separate the two branches in $f(x,\, \mu)$ and express it as:

$$f(x, \mu) = xg(x,\, \mu),$$

where $x = 0$ is one brunch and it is assumed that $g(0,\, \mu) \neq 0$. Next we calculate:

$$f_x(0,\, \mu) = g(0,\, \mu) + xg_x(0,\, \mu) = g(0,\, \mu), \tag{17}$$

which implies from (15)-(16) that

$$g(0,0) = 0,\; g_\mu(0,\, 0) \neq 0.$$

Considering a pitchfork bifurcation in the $x - \mu$ plane, we note that a pitchfork bifurcation occurs if:

$$\mu'(x)|_{x=0} = 0, \quad \frac{d^2\mu}{dx^2}|_{x=0} \neq 0.$$

Using the chain rule we find:

$$0 = \mu'(0) = -\frac{g_x(0,0)}{g_\mu(0,0)} = -\frac{f_{xx}(0,0)}{f_{x\mu}(0,0)}, \tag{18}$$

where we used in the last equality (17). Proceeding similarly we get

$$0 \neq \mu''(0) = -\frac{g_{xx}(0,0)}{g_\mu(0,0)} = -\frac{f_{xxx}(0,0)}{f_{x\mu}(0,0)}. \tag{19}$$

Combining (18)-(19) with (16) we get the following conditions specific for the pitchfork bifurcation:

$$f_\mu(0,0) = f_{xx}(0,0) = 0,$$
$$f_{xxx}(0,0) \neq 0,\;\; f_{x\mu} \neq 0.$$

Q5 (Hopf bifurcation) We study the Brusselator

$$\begin{aligned}\dot{x} &= 1-(b+1)x+x^2y, \quad (20)\\ \dot{y} &= bx-x^2y, \quad (21)\end{aligned}$$

The only fixed point of this system is $(x,y)=(1,b)$. Linearising around it, we find

$$Df(0,\,0)=\begin{bmatrix} b-1 & 1\\ -b & -1\end{bmatrix},$$

so that the corresponding eigenvalues are:

$$\lambda_{\pm}=\frac{b-2\pm\sqrt{b(b-4)}}{2},$$

which suggests that there is a Hopf bifurcation at $b_c=2$ when $\lambda_{\pm}$ are pure imaginary.

We want to rewrite system (20)-(21) into a form such that the fixed point is at the origin and the bifurcation occurs when the bifurcation parameter is zero. To this end, we rescale $\tilde{x}=x-1$, $\tilde{y}=y-b$, and $\beta=(b-2)/2$ and transform (20)-(21) to obtain (upon dropping the tilde symbols):

$$\begin{aligned}\dot{x} &= (2\beta+1)x+y+2xy+(2\beta+2)x^2+x^2y, \quad (22)\\ \dot{y} &= -(2\beta+2)x-y-2xy-(2\beta+2)x^2-x^2y, \quad (23)\end{aligned}$$

In this case, the eigenvalues at the origin are:

$$\lambda_{\pm}(\beta)=\beta\pm i\sqrt{1-\beta^2},$$

with the corresponding eigenvectors:

$$\begin{bmatrix} -1/2\\ 1\end{bmatrix}\pm i\begin{bmatrix} 1/2\\ 0\end{bmatrix}, \quad (24)$$

when calculated at the bifurcation point $\beta=0$.

We now employ the Hopf theorem. Close to the bifurcation for $\beta\ll 1$, we can transform the dynamical system (22)-(23) into the polar form:

$$\begin{aligned}\dot{r} &= d\beta r+ar^3, \quad (25)\\ \dot{\theta} &= \omega+O(r^2), \quad (26)\end{aligned}$$

where

$$d=\frac{d}{d\beta}\mathrm{Re}\lambda_{\pm}|_{\beta=0}=1,\quad \omega=\mathrm{Im}\lambda_{+}(0)=1,$$

and Lyapunov coefficient

$$a = \frac{1}{16w}\{(f_{uuu}+f_{uvv}+g_{uuv}+g_{vvv})w+f_{uv}(f_{uu}+f_{vv})-g_{uv}(g_{uu}+g_{vv})-f_{uu}g_{uu}+f_{vv}g_{vv}\}|_{(u,v)=(0,0)} \tag{27}$$

for new dependent variables u and v, which are a linear transformation of x and y that transform (22)-(23) considered with $\beta =$ into the form:

$$\dot{u} = -v + f(u, v), \tag{28}$$
$$\dot{v} = u + g(u, v). \tag{29}$$

For that create a transformation matrix (whose columns are eigenvectors (24)):

$$P = \begin{bmatrix} -1/2 & 1/2 \\ 1 & 0 \end{bmatrix},$$

with

$$P^{-1} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix},$$

so that

$$\begin{bmatrix} u \\ v \end{bmatrix} = P^{-1} \begin{bmatrix} x \\ y \end{bmatrix}$$

This transformation implies that

$$x = -\frac{u}{2} + \frac{\omega v}{2(1+\beta)}, \tag{30}$$
$$y = u, \tag{31}$$

and

$$\begin{aligned}
\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \dot{y} \\ 2\dot{x} + \dot{y} \end{bmatrix} \\
&= \begin{bmatrix} -2x - y - 2xy - 2x^2 - x^2 y \\ y + 2xy + 2x^2 + x^2 y \end{bmatrix} \text{ (using (22)-(23) with } \beta = 0) \\
&= \begin{bmatrix} -v + \frac{u^2}{2} - \frac{v^2}{2} - \frac{1}{4}(u^2 - 2uv + v^2)u \\ u + \frac{v^2}{2} - \frac{u^2}{2} + \frac{1}{4}(u^2 - 2uv + v^2)u \end{bmatrix} \text{ (using (30)-(31))} .
\end{aligned}$$

Hence, we obtain that system (22)-(23) transform for $\beta = 0$ into (28)-(29) with:

$$f(u, v) = -g(u, v) = -\frac{v^2}{2} + \frac{u^2}{2} - \frac{v^2 u}{4} + \frac{vu^2}{2} - \frac{u^3}{4}.$$

Calculating Lyapunov coefficient a using formula (27) gives

$$a = -\frac{3}{16},$$

so that the Hopf normal form of the system is:

$$\dot{r} = \beta r - \frac{3}{16} r^3, \tag{32}$$

with the leading order period

$$T \sim \frac{2\pi}{\omega} = 2\pi.$$

We note from (32) that $r = 0$ is asymptotically stable for $\beta \leq 0$, but becomes unstable for $\beta > 0$, whilst introducing a new asymptotically stable radius at $r = 4\sqrt{\beta/3}$, suggesting that the Hopf bifurcation is *supercritical*.