Nonlinear Systems-lecture notes 5

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5 Introduction to chaotic dynamical systems

The material of this chapter is covered in the following books:

- L. Perko, *Differential Equations and Dynamical Systems* (Second edition, Springer, 1996). Paragraphs **4.8–4.9**.
- Guckenheimer and Holmes, *Nonlinear Oscillations, Dynamical Systems* (Springer, 1983). Paragraphs **4.5, 5.1–5.3**.
- Y. A. Kuznetsov, *Elements of Applied Bifurcation Theory* (Second edition, Springer, 1998). Paragraphs **1.1.4, 1.3, 6.1, 6.4.1**.

We saw in the example of the logistic map that chaotic dynamics (which arises for $\mu > \mu_\infty$) is characterised by two properties of its solutions : sensitivity to the initial conditions and aperiodic behaviour. In this chapter, we will give a unified mathematical meaning for these notions by considering special examples of chaotic dynamical systems.

5.1 Symbolic dynamics

Symbolic dynamics is a special example of discrete dynamical systems, whose phase space is given by the set of bi-infinite sequences of 0 and 1 of the form

$$s = \{..., s_n, ..., s_{-1}|s_0 s_1 ... s_n ...\} \text{ with } n \in \mathbb{Z} \text{ and } s_n \in \{0,\, 1\}.$$

A natural distance between any two elements $s,\ s' \in \Sigma$ is given by

$$|s - s'| = \operatorname{dist}(s,\, s') = \sum_{n\in\mathbb{Z}} \frac{|s_n - s'_n|}{2^n} < \infty.$$

It is easy to deduce that

$$\text{dist}(s,\, s') < \frac{1}{2^{n-1}} \text{ iff } s_i = s_i' \text{ for all } i \in \{-n, -n+1, ..., 0, ..., n-1, n\}. \quad (1)$$

Consider the *Bernoulli shift map* σ acting in the phase space Σ by the following rule:

$$s = \{...s_{-2}s_{-1}|s_0s_1...\} \in \Sigma \implies \sigma(s) = \{...s_{-2}s_1s_0|s_1s_2...\} \in \Sigma. \quad (2)$$

In other words, $s' = \sigma(s)$ implies $s_i' = s_{i+1}$.

The dynamical system produced by the shift map σ in the phase of bi-infinite sequences Σ is called *symbolic dynamics*. The following theorem provides a characterisation of typical orbits of map σ.

Theorem 1. (i) : *σ has a countable infinity of periodic orbits with an arbitrary period $N \in \mathbb{N}$.*

(ii) : *σ has an uncountable infinity of non-periodic orbits.*

(iii) : *σ has a dense orbit in Σ.*

(iv) : *The dynamics of σ is sensitive to the initial conditions. Namely, for any arbitrary close two initial points $s,\, s' \in \Sigma$ there exists $n \in \mathbb{N}$ such that*

$$|\sigma^n(s) - \sigma^n(s')| \geq 1. \quad (3)$$

Proof: (i) : We can construct a periodic orbit of σ having an arbitrary period $N \in \mathbb{N}$ by first composing N digits of 0 and 1 and then repeating them periodically. An example of a such period-4 orbit is given by

$$\{1010|1010|...\}.$$

(ii) : We can map any $s \in \Sigma$ bijectively to a binary number $s \in (0,\, 1)$ of the form:

$$\tilde{s} = 0.s_0s_1s_{-1}s_2s_{-2}...$$

By taking the preimage under such map of a number $\tilde{s} \in \mathbb{R} \setminus \mathbb{Q}$, where $\mathbb{Q}$ is the set of rational numbers, one obtains the bi-infinity sequence $s \in \Sigma$ such that $\sigma^n(s)$ never repeats itself. Hence, $\tilde{s} \in \mathbb{R} \setminus \mathbb{Q}$ corresponds to non-periodic orbits of σ. As the set $\mathbb{R} \setminus \mathbb{Q}$ is uncountable we can construct an uncountable infinity of such non-periodic orbits.

(iii) : First, we construct $s_0 \in \Sigma$ as a concatenation over $N \in \mathbb{N}$ of all 2^N binary blocks of size N. Then for any $s \in \Sigma$ and any $k \in \mathbb{Z}$ there exists $n \in \mathbb{Z}$ such that

$$|\sigma^n(s_0) - s| < \frac{1}{2^{k-1}}. \quad (4)$$

Indeed, by construction of s_0 one can always find a shift $n \in \mathbb{Z}$ such that

$$[\sigma^n(s_0)]_i = s_i \text{ for all } i \text{ with } |i| \leq k.$$

The last property then implies (4). Hence, we have shown that for any $s \in \Sigma$ there exists an arbitrary closed to it point of orbit $\sigma^n(s_0)$, i.e. s_0 generates a dense orbit in Σ.

(iv) : For any $k \in \mathbb{N}$ consider s, $s' \in \Sigma$ such that $\mathrm{dist}(s, s') < 2^{k-1}$. By (1) sequences s and s' have the same central blocks between registers $-k$ and k. Additionally, there exists $N \geq k+1$ such that $[\sigma^N(s)]_0 \neq [\sigma^N(s')]_0$. But this implies (again using (1)) that $d(\sigma^N(s), \sigma^N(s')) \geq 1$.□

By analysing properties $(i) - (iv)$ of the shift map σ stated in Theorem 1 we can introduce a general notion of the *chaotic set* of a (discrete or continuous) dynamical system.

Definition 5.1. Let f be a bijective map defined on a phase space M. Let Λ be an invariant compact subset of M. Λ is chaotic if the following two properties hold:
(i) : f has sensitivity to the initial data belonging to Λ.
(ii) : f is *topologically transitive* in Λ, i.e. for all open sets U, $V \in \Lambda$ there exists $n \in \mathbb{Z}$ such that $f^n(U) \cap V \neq \emptyset$.

Remark 5.1. It is easy to show that if Λ has a dense orbit in M then f is topologically transitive.

Remark 5.2. By Theorem 1 and the last remark the shift map σ acting in Σ has the chaotic set $\Lambda = \Sigma$.

In the next paragraph, we will present an example of chaotic sets appearing in the special non-autonomous two-dimensional continuous dynamical systems.

5.2 Chaos in periodically perturbed planar systems

Melnikov's method[1] is one among only few existing methods capable for identification of chaotic sets and analytical investigation of their structure. In this paragraph, we first present Melnikov method for planar autonomous Hamiltonian systems and then extend it to the special non-autonomous ones with chaotic sets the dynamics in which, interestingly, turns out to be isomorphic to the symbolic dynamics discussed in the previous paragraph.

[1]Melnikov, V. On the stability of the center for time periodic perturbations, *Trans. Moscow Math. Soc.* 12:1-57, 1963.

5.2.1 Melnikov's method for planar autonomous systems

Consider system

$$\dot{x} = f(x, \varepsilon) \text{ with } x \in \mathbb{R}^2 \text{ and } \varepsilon \geq 0. \tag{5}$$

In the extended phase-space $(x,\, \varepsilon)$, it can be written as

$$\begin{aligned} \dot{x} &= f(x,\, \varepsilon), \\ \dot{\varepsilon} &= 0. \end{aligned} \tag{6}$$

Assume that system (5) has a homoclinic orbit $\bar{x}_0(t)$ for $\varepsilon = 0$, i.e. the solution $\bar{x}_0(t)$ such that

$$\lim_{t \to \pm\infty} \bar{x}_0(t) = x_0,$$

where x_0 is a saddle point. We assume also that for sufficiently small $\varepsilon > 0$ there exists a curve of perturbed saddles x_ε and denote the corresponding global stable and unstable manifolds by $W^u(x_\varepsilon)$ and $W^s(x_\varepsilon)$. Let us denote also the corresponding unions of these manifolds in the extended phase-space by $\mathcal{W}^u$ and $\mathcal{W}^s$, respectively. We would like to find an analytical condition for the extended manifolds $\mathcal{W}^u$ and $\mathcal{W}^s$ to intersect *transversely*. This means that two extended manifolds intersect only along the homoclinic orbit $x_0(t)$, i.e. $\mathcal{W}^u \cap \mathcal{W}^s = x_0(t)$, and then split for $\varepsilon > 0$ (see Fig 1). The transversal intersection also implies that $W^u(x_0) = W^s(x_0)$ and for all $t \in \mathbb{R}$ the vector $\xi(t) = \dot{\bar{x}}(t)$ is tangent both to $\mathcal{W}^u$ and $\mathcal{W}^s$, so that

$$T_{(\bar{x}_0(t),\, 0)}\mathcal{W}^s \cap T_{(\bar{x}_0(t),\, 0)}\mathcal{W}^u = \operatorname{span}\{\xi(t)\}.$$

Because of that, $u(t) = (\xi(t), 0)^T$ is a unique bounded for all $t \in \mathbb{R}$ solution to the linearisation of system (6) at $(\bar{x}_0(t),\, 0)^T$:

$$\begin{aligned} \dot{u} &= f_x(x_0(t),\, 0)u + f_\varepsilon(x_0(t),\, 0)\varepsilon, \\ \dot{\varepsilon} &= 0. \end{aligned} \tag{7}$$

By introducing the matrix-valued function

$$A(t) = f_x(x_0(t),\, 0),$$

one observes also that $\xi(t)$ is a unique bounded solution to the so-called *variational equation* $\dot{u} = A(t)u$.

One can also show that there exists a unique bounded solution $\eta(t)$ to the *adjoint variational equation*:

$$\dot{v} = -A^T(t)v, \quad v \in \mathbb{R}^n. \tag{8}$$

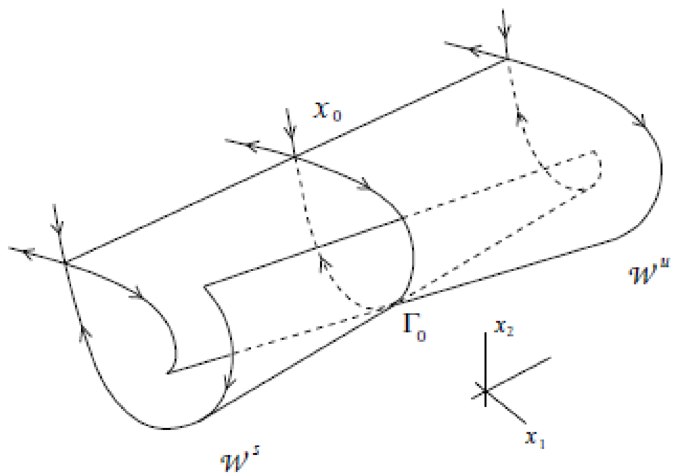

Figure 1: Transversal intersection of the extended global stable $\mathcal{W}^s$ and unstable $\mathcal{W}^u$ manifolds.

Moreover, by calculating

$$\begin{aligned}\frac{d}{dt} < \eta,\, \xi > &= < \dot{\eta},\, \xi > + < \eta,\, \dot{\xi} > \\ &= - < A^T\eta,\, \xi > + < \eta,\, A\xi >= - < \eta,\, A\xi > + < \eta,\, A\xi >= 0,\end{aligned}$$

one obtains that

$$< \eta,\, \xi >= \text{const } = 0, \tag{9}$$

where we have used that $\xi(t) = \dot{\bar{x}}(t) \to 0$ as $t \to \pm\infty$. Hence, we deduced that at any point $\bar{x}_0(t)$ of the unperturbed homoclinic orbit vectors $\xi(t)$ and $\eta(t)$ span the tangential and normal directions to it, respectively (see Fig. 2). Now, we can deduce the condition suggested by Melnikov for checking that manifolds $\mathcal{W}^s$ and $\mathcal{W}^u$ intersect transversely. Suppose the opposite, then there should exists another solution $\xi_1(t) \neq \xi(t)$ to system (7) which is tangential to both manifolds $\mathcal{W}^s$ and $\mathcal{W}^u$.

Let us take the scalar product of $\eta(t)$ and first equation in (7) considered

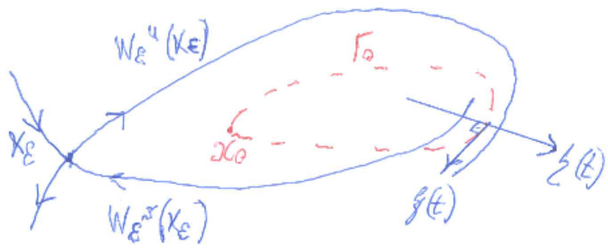

Figure 2: Plot of the homoclinic orbit $\bar{x}_0(t)$ and perturbed split manifolds $W^s(x_\varepsilon)$ and $W^u(x_\varepsilon)$ for $0 < \varepsilon \ll 1$.

with $u(t) = \xi_1(t)$:

$$
\begin{aligned}
\varepsilon \int_{-\infty}^{\infty} < \eta(t),\, f_\varepsilon(\bar{x}_0(t),\, 0) > \, dt &= \int_{-\infty}^{\infty} < \eta(t),\, \dot{\xi}_1(t) - A(t)\xi_1(t) > \, dt \\
&= \; < \eta(t),\, \xi_1(t) > |_{-\infty}^{+\infty} - \int_{-\infty}^{\infty} < \dot{\eta}(t) + A^T(t)\eta(t),\, \xi_1(t) > \, dt \\
&= 0, \qquad (10)
\end{aligned}
$$

where we used that

$$< \eta(t),\, \xi_1(t) > (\pm\infty) = 0,$$

because $\eta(t)$ is bounded and $\xi_1(t) \to 0$ as $t \to \pm\infty$.

We conclude from (10) that the integral

$$M = \int_{-\infty}^{\infty} < \eta(t),\, f_\varepsilon(\bar{x}_0(t),\, 0) > \, dt = 0 \qquad (11)$$

necessarily if $\mathcal{W}^s$ and $\mathcal{W}^u$ intersect non-transversely. The integral M is called Melnikov integral.

For two-dimensional systems (5) with $x = (x_1,\, x_2)^T$ function $\eta(t)$ can be found explicitly and Melnikov integral can be written solely in terms of function $f(x_1, x_2, \varepsilon) = (f_1(x_1,\, x_2,\, eps),\, f_2(x_1,\, x_2,\, \varepsilon)^T$.

Lemma 1. *Integral (11) can be written as*

$$M = \int_{-\infty}^{+\infty} \exp\left[- \int_0^t \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) d\tau \right] \left(f_1 \frac{\partial f_2}{\partial \varepsilon} - f_2 \frac{\partial f_1}{\partial \varepsilon} \right) dt. \qquad (12)$$

Moreover, consider the special case when (5) is a perturbation of a Hamiltonian system, i.e.

$$\dot{x} = f(x) + \varepsilon g(x), \tag{13}$$

with $f(x_1, x_2) = (H_{x_1}, -H_{x_2})^T$ *and function* $H(u, v) \in C^2(\mathbb{R}^2, \mathbb{R})$ *being the first integral of the unperturbed system:*

$$\dot{x}_1 = \frac{\partial H}{\partial x_1}, \quad \dot{x}_2 = -\frac{\partial H}{\partial x_2}.$$

Then for (13) formula (12) can be reduced further to

$$M = \int_{-\infty}^{+\infty} f(\bar{x}_0(t)) \wedge g(\bar{x}_0(t))\, dt, \tag{14}$$

where the wedge product of two vectors $f = (f_1, f_2)^T$ *and* $g = (g_1, g_2)^T$ *is defined by*

$$f \wedge g = f_1 g_2 - f_2 g_1 \in \mathbb{R}.$$

Proof: To prove (12) we use

$$\dot{x}_0(t) = \begin{bmatrix} f_1(x_0(t), 0) \\ f_2(x_0(t), 0) \end{bmatrix}$$

and (9) to write

$$\eta(t) = \varphi(t) \begin{bmatrix} -f_2(x_0(t), 0) \\ f_1(x_0(t), 0) \end{bmatrix}, \tag{15}$$

with $\varphi(t) : \mathbb{R} \to \mathbb{R}$. To determine function $\varphi(t)$ we use that $\eta(t)$ solves the adjoint variational equation (8), i.e.

$$\dot{\eta} = -\begin{bmatrix} f_{1,x} & f_{2,x} \\ f_{1,y} & f_{2,y} \end{bmatrix} \eta.$$

By substituting (15) into the last equation we deduce that $\varphi(t)$ solves

$$\dot{\varphi}(t) = -(f_{1,x} + f_{2,x})\varphi(t).$$

Solving the last equation one obtains

$$\varphi(t) = \exp\left[-\int_0^t \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right) d\tau\right].$$

Now, substitution of the formula for $\varphi(t)$ into (15) and then into (11) transforms the latter into (12).

In the case of the perturbed Hamiltonian system (13), one can explicitly calculate that

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = \frac{\partial H}{\partial x \partial y} - \frac{\partial H}{\partial y \partial x} = 0,$$
$$\frac{\partial f_1}{\partial \varepsilon} = g_1, \ \frac{\partial f_2}{\partial \varepsilon} = g_2.$$

Using the latter expressions we reduce (12) to a simpler formula (14).□

We summarise that if the invariant manifolds $\mathcal{W}^s$ and $\mathcal{W}^u$ to the extended system (6), which has the homoclinic orbit $\bar{x}_0(t)$ for $\varepsilon = 0$, intersect tangentially then the Melnikov integral (12) is necessarily zero. In this case, there exist special right-hand sides $f(x, \varepsilon)$ for which the perturbed homoclinic orbit $\bar{x}_\varepsilon(t)$ persists for all sufficiently small ε and, therefore, $W^s_\varepsilon(x_\varepsilon) = W^u_\varepsilon(x_\varepsilon)$.

The following theorem provides sufficient conditions for this to be true in the case of the perturbed Hamiltonian systems.

Theorem 2. *Consider system*

$$\dot{x} = f(x) + \varepsilon g(x, \mu) \text{ with } x \in \mathbb{R}^2, \ \varepsilon > 0, \ \mu \in \mathbb{R}, \tag{16}$$

and $f(x_1, x_2) = (H_{x_1}, -H_{x_2})^T$ *for some function* $H(x_1, x_2) \in C^2(\mathbb{R}^2, \mathbb{R})$. *Suppose that for* $\varepsilon = 0$ *there exists a homoclinic orbit* $\bar{x}_0(t)$ *of (16). Define according to (14) the Melnikov integral:*

$$M(\mu) = \int_{-\infty}^{+\infty} f(\bar{x}_0(t)) \wedge g(\bar{x}_0(t), \mu)\, dt. \tag{17}$$

If there exists $\mu_0 \in \mathbb{R}$ *with*

$$M(\mu_0) = 0 \text{ and } M'(\mu_0) \neq 0, \tag{18}$$

then for any sufficiently small $\varepsilon > 0$ *there exists* $\mu_\varepsilon = \mu_0 + O(\varepsilon)$ *such that, system (16) considered with parameters* $(\mu_\varepsilon, \varepsilon)$ *has a unique homoclinic orbit* $\bar{x}_\varepsilon(t)$, *such that*

$$\operatorname{dist}(\bar{x}_0(t), \bar{x}_\varepsilon(t)) \to 0 \text{ as } \varepsilon \to 0.$$

5.2.2 Melnikov's method for periodically perturbed planar systems

Consider system

$$\dot{x} = f(x) + \varepsilon g(x, t) \text{ with } \varepsilon \geq 0, \tag{19}$$

$f(x_1, x_2) = (H_{x_1}, -H_{x_2})^T$ and $g(x_1, x_2, t+T) = g(x_1, x_2, t)$ for $x = (x_1, x_2)^T \in \mathbb{R}^2$ and $t \in S^1$. Here torus S^1 is the period interval $[0,\, T)$ of the perturbation function $g(x, t)$ with the end points 0 and T being identified.

As in previous paragraph, we assume that the unperturbed Hamiltonian system

$$\dot{x} = f(x)$$

has a homoclinic orbit $\bar{x}_0(t)$ connecting the saddle point x_0 with itself. But now system (19) is rather non-autonomous. Using the implicit function theorem one can show that for sufficiently small $\varepsilon > 0$ the extended system

$$\begin{aligned} \dot{x} &= f(x) + \varepsilon g(x, t), \\ \dot{t} &= 0, \end{aligned} \tag{20}$$

will have a unique periodic solution $\gamma_\varepsilon(t) = (x_\varepsilon^t,\, t)$, such that

$$x_\varepsilon^t = x_0 + x_{1,\varepsilon}^t \text{ with } |x_{1,\varepsilon}^t| \leq C\varepsilon \text{ for } t \in [0,\, T).$$

Let us define for each $t \in [0, T)$ the Poincaré map $P_\varepsilon^{t_0} : \Sigma^{t_0} \to \Sigma^{t_0}$ with the corresponding transversal section Σ^{t_0} to the orbits of (20):

$$\Sigma^{t_0} = \{(x, t) |\, t = t_0 \in [0,\, T)\} \subset \mathbb{R}^2 \times S^1.$$

Note that the periodic solution $\gamma_\varepsilon(t)$ induces the fixed point $x_\varepsilon^{t_0} \in \Sigma^{t_0}$ of $P_\varepsilon^{t_0}$. Next, we define the global stable $\mathcal{W}_\varepsilon^s(\gamma_\varepsilon(t))$ and unstable $\mathcal{W}_\varepsilon^s(\gamma_\varepsilon(t))$ manifolds as unions of points (x, t_{in}) in the extended phase-space $\mathbb{R}^2 \times S^1$ such that

$$\text{dist}\left(\varphi_\varepsilon^t(x,\, t_{in}),\, x_\varepsilon^t\right) \to 0$$

as $t \to +\infty$ and $t \to -\infty$, respectively. Here, $\varphi_\varepsilon^t(x,\, t_{in}) \in \mathbb{R}^2$ denotes the solution to system (19) considered with the initial data $(x,\, t_{in})$. Correspondingly, let us denote by

$$W_\varepsilon^{u,s}(x_\varepsilon^{t_0}) = \mathcal{W}_\varepsilon^{u,s}(\gamma_\varepsilon(t)) \cap \Sigma^{t_0}$$

the unstable/stable manifolds at the fixed point $x_\varepsilon^{t_0}$ of the map $P_\varepsilon^{t_0}$ (see Fig. 3).

For each fixed $\varepsilon > 0$ and $t_0 \in [0, T)$, these two manifolds can intersect or be disjoint depending on the particular perturbation $g(x, t)$ considered in (19). The *Melnikov's method* presented here provides a simple analytical criterion whether $W_\varepsilon^u(x_\varepsilon^{t_0}) \cap W_\varepsilon^s(x_\varepsilon^{t_0}) = \emptyset$ for some $\varepsilon > 0$ and $t_0 \in [0,\, T)$ or not.

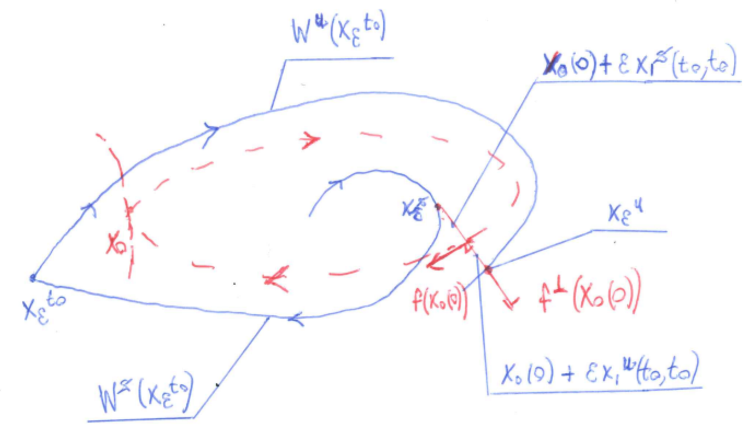

Figure 3: Sketch of the invariant manifolds $W_\varepsilon^u(x_\varepsilon^{t_0})$ and $W_\varepsilon^s(x_\varepsilon^{t_0})$ in the section Σ^{t_0} of the Poincaré map $P_\varepsilon^{t_0}$.

First, observe that

$$\operatorname{dist}\left(\mathcal{W}_\varepsilon^{s,u}(\gamma_\varepsilon(t)),\, \bar{x}_0(t)\right) \to 0 \text{ as } \varepsilon \to 0$$

uniformly in the extended phase-space $\mathbb{R}^2 \times S^1$ provided function $g(x,t)$: $\mathbb{R}^2 \times S^1 \to \mathbb{R}$ is sufficiently smooth. Correspondingly, for a fixed $\bar{x}_0(0)$ there are two points $x_\varepsilon^{s,u} \in W_\varepsilon^{u,s}(x_\varepsilon^{t_0})$ such that,

$$|x_\varepsilon^{s,u} - \bar{x}_0| = O(\varepsilon) \text{ as } \varepsilon \to 0.$$

Consequently, two solutions $\varphi_\varepsilon^t(x_\varepsilon^s, t_0)$ to (19) generated by points $x_\varepsilon^{s,u}$ can be expanded uniformly in time as

$$\begin{aligned}
\varphi_\varepsilon^t(x_\varepsilon^s, t_0) &= \bar{x}_0(t-t_0) + \varepsilon x_1^s(t,t_0) + O(\varepsilon^2) \text{ for } t \in (t_0, +\infty), \\
\varphi_\varepsilon^t(x_\varepsilon^u, t_0) &= \bar{x}_0(t-t_0) + \varepsilon x_1^u(t,t_0) + O(\varepsilon^2) \text{ for } t \in (-\infty, t_0),
\end{aligned} \tag{21}$$

with $x_1^{s,u}(t,t_0)$ being solutions to the linearized equations

$$\begin{aligned}
\dot{x}_1^s(t,t_0) &= f_x(\bar{x}_0(t-t_0))x_1^s(t,t_0) + g(\bar{x}_0(t-t_0),t) \text{ for } t \geq t_0, \\
\dot{x}_1^u(t,t_0) &= f_x(\bar{x}_0(t-t_0))x_1^u(t,t_0) + g(\bar{x}_0(t-t_0),t) \text{ for } t \leq t_0.
\end{aligned} \tag{22}$$

Next, let us introduce the distance vector (see Fig. 3)

$$d(t_0) = x_\varepsilon^u - x_\varepsilon^s.$$

From (21) estimated at $t = t_0$ one obtains the corresponding expansion

$$d(t_0) = \varepsilon \frac{f(\bar{x}_0(0)) \wedge (x_1^u(t_0,\, t_0) - x_1^s(t_0,\, t_0))}{|f(\bar{x}_0(0))|} + O(\varepsilon^2). \tag{23}$$

Note that in the above formula $f(\bar{x}_0(0)) \wedge (x_1^u(t_0, t_0) - x_1^s(t_0, t_0))$ is the projection of $x_1^u(t_0, t_0) - x_1^s(t_0, t_0)$ onto the normal

$$f^{\perp}(\bar{x}_0(0)) = (-f_2(\bar{x}_0(0)),\; f_1(\bar{x}_0(0)))^T$$

to the homoclinic orbit $\bar{x}_0(t)$ at the point $\bar{x}_0(0)$.

Finally, let us define the *Melnikov integral* for this problem as

$$M(t_0) = \int_{-\infty}^{+\infty} f(\bar{x}_0(t - t_0)) \wedge g(\bar{x}_0(t - t_0), t)\, dt. \tag{24}$$

We can now state and prove the following theorem.

Theorem 3. *If function $M(t)$ has a simple zero at $t = t_0$ then $W_\varepsilon^u(x_\varepsilon^{t_0})$ and $W_\varepsilon^s(x_\varepsilon^{t_0})$ intersect transversely for all sufficiently small $\varepsilon > 0$. If $M(t) \neq 0$ for all $t \in [0,\, T)$ then $W_\varepsilon^u(x_\varepsilon^{t_0}) \cap W_\varepsilon^s(x_\varepsilon^{t_0}) = \emptyset$ for all sufficiently small $\varepsilon > 0$ and $t \in [0,\, T)$.*

Proof: Let us introduce functions

$$\begin{aligned} \Delta^{s,u}(t,\, t_0) &= f(\bar{x}_0(t - t_0)) \wedge x_1^{s,u}(t,\, t_0), \\ \Delta(t,\, t_0) &= \Delta^u(t,\, t_0) - \Delta^s(t,\, t_0). \end{aligned}$$

Due to (23) one has

$$d(t_0) = \varepsilon \frac{\Delta(t_0, t_0)}{|f(\bar{x}_0(0))|} + O(\varepsilon^2). \tag{25}$$

Calculate the derivative,

$$\frac{d}{dt}\Delta^s(t,\, t_0) = f_x(\bar{x}_0(t - t_0))\dot{\bar{x}}_0(t - t_0) \wedge x_1^s(t,\, t_0) + f(\bar{x}_0(t - t_0)) \wedge \left(\frac{\partial}{\partial t} x_1^s(t,\, t_0)\right).$$

Substitution of $\dot{\bar{x}}_0(t - t_0) = f(\bar{x}_0(t - t_0))$ and equation (22) into the last relation gives

$$\frac{d}{dt}\Delta^s(t,\, t_0) = \operatorname{tr} f_x(\bar{x}_0(t - t_0))\Delta^s(t,\, t_0) + f(\bar{x}_0(t - t_0)) \wedge g(\bar{x}_0(t - t_0), t) \tag{26}$$

Using the fact that the unperturbed system is Hamiltonian ($f(x_1, x_2) = (H_{x_1}, -H_{x_2})^T$) one calculates that $\mathrm{tr}\, f_x(\bar{x}_0(t-t_0)) = 0$ for all $t \in \mathbb{R}$. Therefore, (26) reduces to

$$\frac{d}{dt}\Delta^s(t,\, t_0) = f(\bar{x}_0(t-t_0)) \wedge g(\bar{x}_0(t-t_0), t).$$

Integration of the last expression between t_0 and $+\infty$ gives

$$\Delta^s(+\infty,\, t_0) - \Delta^s(t_0,\, t_0) = \int_{t_0}^{\infty} f(\bar{x}_0(t-t_0)) \wedge g(\bar{x}_0(t-t_0), t)\, dt, \qquad (27)$$

where

$$\Delta^s(+\infty,\, t_0) = \lim_{t \to +\infty} f(\bar{x}_0(t-t_0)) \wedge x_1^s(t,\, t_0).$$

Note that $\Delta^s(+\infty,\, t_0) = 0$, because $x_1^s(t,\, t_0)$ is bounded for all $t \geq t_0$ and

$$\lim_{t \to +\infty} f(\bar{x}_0(t-t_0)) = \lim_{t \to +\infty} \dot{\bar{x}}_0(t-t_0) = 0$$

by definition of the homoclinic orbit $\bar{x}_0(t)$.

Therefore, (27) implies that

$$\Delta^s(t_0,\, t_0) = -\int_{t_0}^{\infty} f(\bar{x}_0(t-t_0)) \wedge g(\bar{x}_0(t-t_0), t)\, dt. \qquad (28)$$

Similar arguments give that

$$\Delta^u(t_0,\, t_0) = \int_{-\infty}^{t_0} f(\bar{x}_0(t-t_0)) \wedge g(\bar{x}_0(t-t_0), t)\, dt, \qquad (29)$$

Through substitution of (28)-(29) into (25) we conclude that

$$d(t_0) = \varepsilon \frac{M(t_0)}{|f(\bar{x}_0(0))|} + O(\varepsilon^2) \qquad (30)$$

with Melnikov integral $M(t_0)$ defined in (24).

Now, in order to prove the statements of the theorem we observe that for sufficiently small $\varepsilon > 0$

$$\mathrm{sign}\, M(t_0) = \mathrm{sign}\, d(t_0).$$

Hence, if $M(t_0)$ has a simple zero at $t_0 \in [0,\, T)$ than function $d(t)$ has it also at t_0. Moreover, a simple zero of $d(t)$ means that manifolds $W_\varepsilon^u(x_\varepsilon^{t_0})$ and $W_\varepsilon^s(x_\varepsilon^{t_0})$ intersect transversely. On contrary, if $M(t)$ stays positive or negative for all $t \in [0,\, T)$ manifolds $W_\varepsilon^u(x_\varepsilon^{t_0})$ and $W_\varepsilon^s(x_\varepsilon^{t_0})$ are disjoint.□

Remark 5.3. By applying the change of variables $t \to t + t_0$ in (24) the Melnikov integral takes a more suitable for practical calculations form:

$$M(t_0) = \int_{-\infty}^{+\infty} f(\bar{x}_0(t)) \wedge g(\bar{x}_0(t), t + t_0)\, dt. \tag{31}$$

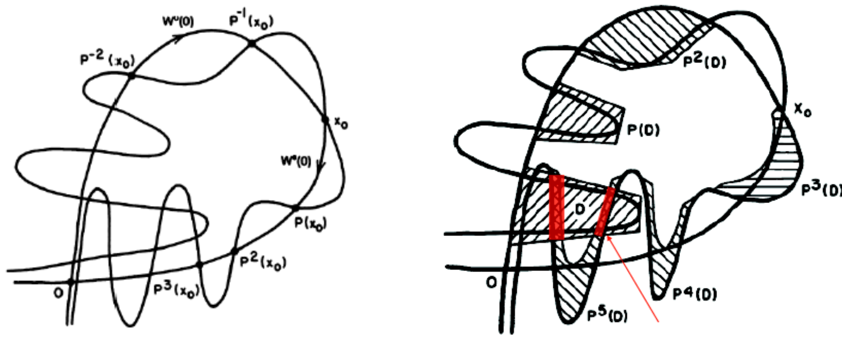

Figure 4: Transversal intersection of the invariant manifolds $W^u_\varepsilon(x^{t_0}_\varepsilon)$ and $W^s_\varepsilon(x^{t_0}_\varepsilon)$ in the section Σ^{t_0} of the Poincaré map $P^{t_0}_\varepsilon$ (left) and intersection of domain D and $P^5(D)$ depicted in red (right).

The analytical statements involving the Melnikov integral in the previous Theorem have a direct implication about existence of *chaotic sets* (in the sense of Definition 5.1 in paragraph 5.1) for the dynamical system (19). Namely, if $M(t_0) = 0$ for some $t_0 \in [0, T)$ then there exists at least one point $\tilde{x}_\varepsilon \in W^u_\varepsilon(x^{t_0}_\varepsilon) \cap W^s_\varepsilon(x^{t_0}_\varepsilon)$. This point is not unique, because any iteration of the Poincaré map $[P^{t_0}_\varepsilon]^n(\tilde{x}_\varepsilon)$ will still belong to the intersection of these two manifolds. Hence, there is a countable infinite sequence of points (see Fig. 4(a))

$$\tilde{x}^n_\varepsilon \in W^u_\varepsilon(x^{t_0}_\varepsilon) \cap W^s_\varepsilon(x^{t_0}_\varepsilon) \text{ with } n \in \mathbb{Z}\,.$$

If, additionally,

$$\lim_{n \to +\infty} \tilde{x}^n_\varepsilon = x^{t_0}_\varepsilon,$$

i.e. the sequence $\{\tilde{x}^n_\varepsilon\}$ accumulates at the fixed point $x^{t_0}_\varepsilon$ then the following theorem about existence of the chaotic subset in $\mathbb{R}^2$ holds with the notation $P = P^{t_0}_\varepsilon$ and $x = x^{t_0}_\varepsilon$.

Theorem 4 (Smale-Birkhoff theorem)**.** *Let $P : \mathbb{R}^2 \to \mathbb{R}^2$ be a diffeomorphism such that P has a hyperbolic fixed point x and $W^s(x)$ intersect $W^u(x)$*

transversely. Then there exists $N \in \mathbb{N}$ and an open set $D \in \mathbb{R}^2$ such that the map $f = P^N$ has a compact invariant chaotic set

$$\Lambda = \cap_{n \in \mathbb{Z}} f^n(D) \tag{32}$$

in which f is topologically equivalent to the Bernoulli shift map σ acting in the phase space of bi-infinite sequences Σ (defined and considered in paragraph 5.1).

In Fig. 4(b) an example of such domain D is given. In this case, D is an open set confined between two intersection points of $W^u_\varepsilon(x^{t_0}_\varepsilon)$ and $W^s_\varepsilon(x^{t_0}_\varepsilon)$. One observes that the fifth iteration of Poincaré map $f = P^5(D)$ intersects D again. By iterating maps $f^n(D)$ and $f^{-n}(D)$ for each $n \in \mathbb{N}$ and taking intersections of their images we construct the required set (32). The proof of the fact that thus constructed Λ is the *chaotic set* and of its topological equivalence to the symbolic dynamics relies on the properties of another map–*Smale's horseshoe map*. Investigation of this map will be conducted in the next paragraph.

In the rest of this paragraph, an example of application of Theorems 3 to a system of the form (19) is presented.

Example 5.1. Consider the Duffing equation

$$\ddot{x} = x - x^3 - 2.5\varepsilon\dot{x} \text{ for } x \in \mathbb{R} \text{ and } 0 < \varepsilon \ll 1, \tag{33}$$

describing the motion of the nonlinear oscillator with the linear damping (friction) term $-2.5\varepsilon\dot{x}$. For $\varepsilon = 0$, (35)is a Hamiltonian system with Hamiltonian:

$$H(x, \dot{x}) = H(x, y) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4}.$$

In this case, there is a saddle at the origin, centers at $(\pm 1, 0)$, and two symmetric homoclinic orbits

$$(\bar{x}(t), \bar{y}^{\pm}(t)) = \pm(\sqrt{2}\operatorname{sech} t, -\sqrt{2}\operatorname{sech} t \tanh t)^T. \tag{34}$$

Phase-plane portraits for (35) are shown in Fig. 5. Let us add the periodic perturbation $\varepsilon\mu\cos t$ describing a periodic applied external force to the oscillator and characterised by another parameter $\mu > 0$ (called forcing amplitude). In this case, the dynamical system takes the form:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x - x^3 + \varepsilon(\mu\cos t - 2.5y). \end{aligned} \tag{35}$$

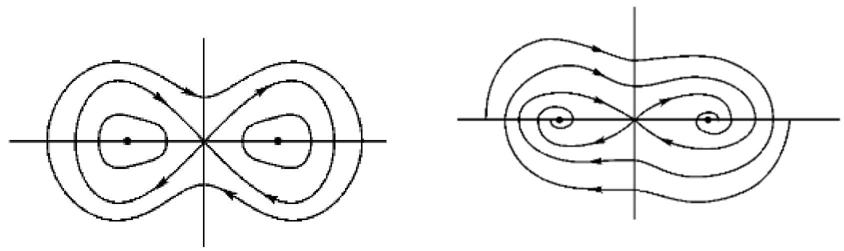

Figure 5: Sketch of the phase-plane portrait for equation (33) considered with $\varepsilon = 0$ (left) and $0 < \varepsilon \ll 1$ (right).

Let us compute the Melnikov function for $(\bar{x}, \bar{y}^+)$ from (34) (computation for $(\bar{x}, \bar{y}^-)$ is identical). According to (31), one has:

$$f(x, y) = \begin{bmatrix} y \\ x - x^3 \end{bmatrix}, \quad g(x, y) = \begin{bmatrix} 0 \\ (\mu \cos t - 2.5y) \end{bmatrix}$$

and, therefore,

$$\begin{aligned} M(t_0) &= \int_{-\infty}^{+\infty} f(\bar{x}(t), \bar{y}^+(t)) \wedge g(\bar{x}(t), \bar{y}^+(t), t + t_0)\, dt \\ &= \int_{-\infty}^{+\infty} \bar{y}^+(t)(\mu \cos(t + t_0) - 2.5\bar{y}^+(t))\, dt. \end{aligned} \tag{36}$$

Substituting (35) into (36) one calculates:

$$\begin{aligned} M(t_0) &= -\sqrt{2}\mu \int_{-\infty}^{\infty} \operatorname{sech} t \tanh t \cos(t + t_0)\, dt - 5 \int_{-\infty}^{\infty} \operatorname{sech}^2 t \tanh^2 t\, dt \\ &= \sqrt{2}\mu\pi \operatorname{sech}(\pi/2) \sin t_0 - \frac{10}{3}, \end{aligned}$$

where we used (without proof) that

$$\int_{-\infty}^{\infty} \operatorname{sech} t \tanh t \cos(t + t_0)\, dt = -\pi \operatorname{sech}(\pi/2) \sin t_0.$$

By calculating $k_0 = 10 \cosh(\pi/2)/(3\sqrt{2}\pi) \approx 1.88$ we, therefore, obtain

$$M(t_0) = \sqrt{2}\mu\pi \operatorname{sech}(\pi/2) \left[\sin(t_0) - \frac{k_0}{\mu} \right]. \tag{37}$$

We observe that for $0 < \mu < k_0$ the Melnikov integral $M(t_0)$ has no zeros and, therefore, by Theorem 3 one has $W_\varepsilon^u(x_\varepsilon^t, y_\varepsilon^t) \cap W_\varepsilon^s(x_\varepsilon^t, y_\varepsilon^t) = \emptyset$ for all

$t \in [0, 1)$. Here $(x_\varepsilon^t,\, y_\varepsilon^t)$ denotes the hyperbolic periodic orbit to system (35) such that

$$\lim_{\varepsilon \to 0} x_\varepsilon^t = \lim_{\varepsilon \to 0} y_\varepsilon^t = 0.$$

In turn, for $\mu > k_0$ manifolds $W_\varepsilon^u(x_\varepsilon^t, y_\varepsilon^t)$ and $W_\varepsilon^s(x_\varepsilon^t, y_\varepsilon^t)$ intersect transversely in countably infinite number of points, see Fig. 6.

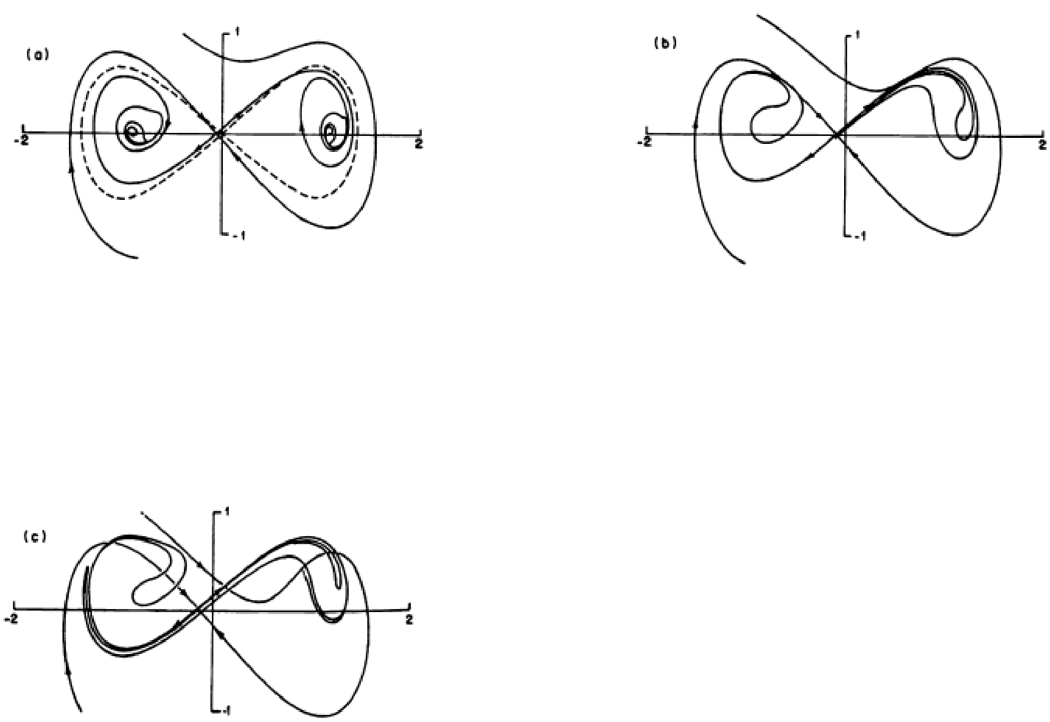

Figure 6: Manifolds $W_\varepsilon^u(x_\varepsilon^0, y_\varepsilon^0)$ and $W_\varepsilon^s(x_\varepsilon^0, y_\varepsilon^0)$ calculated numerically for the Poincare map P_ε^0 of the perturbed Duffing equation (35) for $\varepsilon = 0.1$ and $\mu = 1.1$ (a), $\mu = 1.9$ (b) and $\mu = 3.0$ (c). In the latter case, the manifolds intersect transversely at infinitely many points.

5.3 Smale's horseshoe map

Consider the following map $f : S = [0, 1]^2 \to \mathbb{R}^2$ defined over the unit square. First, S is compressed in the horizontal direction by a factor $0 < \lambda < 1/2$ and stretched by another factor $\mu > 2$ in the vertical direction. The resulting object then is folded in the middle and placed in $\mathbb{R}^2$ so that it intersects the original square S along two vertical strips (see Fig. 7). One

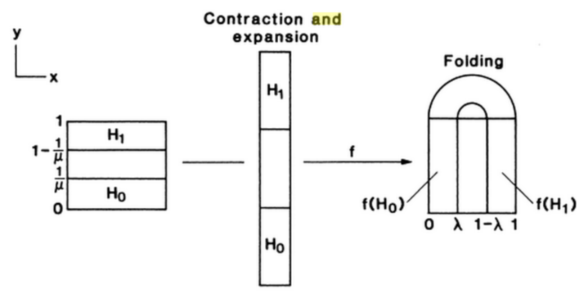

Figure 7: Sketch of the action of the horseshoe map f in S.

observes that the such constructed f maps two horizontal stripes $H_0 = [0,, 1] \times [0, 1/\mu]$ and $H_1 = [0, 1] \times [1 - 1/\mu, 1]$ into two vertical stripes $[0, \lambda] \times [0, 1]$ and $[1 - \lambda, 1] \times [0, 1]$, respectively. Moreover, restricted to H_0 and H_1 the map f is linear and, therefore, one-to-one. Indeed, in H_0

$$f : \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \tag{38}$$

and in H_1

$$f : \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{pmatrix} -\lambda & 0 \\ 0 & -\mu \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ \mu \end{bmatrix}. \tag{39}$$

On the other hand, the set $f(S \setminus (H_0 \cap H_1)) \cap S = \emptyset$.

Let us denote $V_i = f(H_i)$, $i = 1, 2$ and look at the second iteration f^2 in S (see Fig. 8). One observes that f maps $V_1 \cap V_2$ into the four vertical stripes $V_{i_1 i_2}$, $i, j \in \{1, 2\}$ such that $f^{-1}(V_{i_1 i_2}) = H_{i_1} \cup V_{i_2}$. Looking at the higher iterations of f one observes that $f^n(S)$, $n \in \mathbb{N}$ is given by 2^n disjoint vertical stripes $V_{i_1 i_2 \ldots i_n}$ having the width λ^n each. Here, for each $k \in \{1, 2\ldots, n\}$ one has $i_k = 1$ ($i_k = 2$) if $f^{-k}(V_{i_1 i_2 \ldots i_n}) \subset H_0$ ($f^{-k}(V_{i_1 i_2 \ldots i_n}) \subset H_1$).

Analogously, the image of the inverse maps $f^{-n}(S)$, $n \in \mathbb{N}$ is given by 2^n disjoint horizontal stripes $H_{i_1 i_2 \ldots i_n}$ having the width $1/\mu^n$ each. Here,

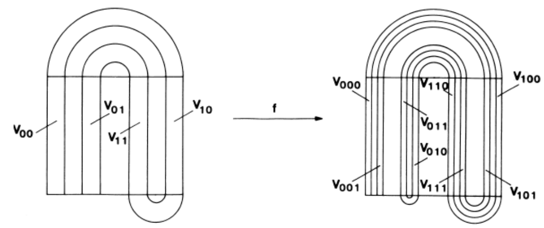

Figure 8: Image of $f^2(S)$ (left) and $f^3(S)$ (right).

for each $k \in \{1, 2..., n\}$ one has $i_k = 1$ ($i_k = 2$) if $f^k(H_{i_1 i_2 \ldots i_n}) \subset H_0$ ($f^k(H_{i_1 i_2 \ldots i_n}) \subset H_1$) (see Fig. 9). Now, if we take for a fixed $n \in \mathbb{N}$ an inter-

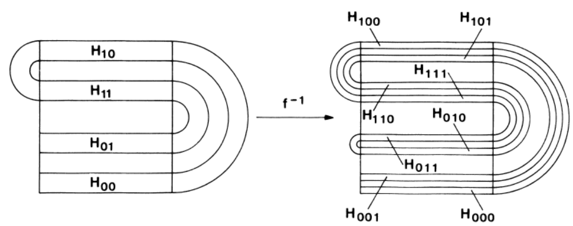

Figure 9: Image of $f^{-2}(S)$ (left) and $f^{-3}(S)$ (right).

section $f^{-n}(S) \cup f^{-n+1}(S) \cap .. \cap f^1(S) \cap f^2(S) \cap f^{n-1}(S) \cap f^n(S)$ it is given by 2^{2n} rectangles with size $(\lambda^n,\ 1/\mu^n)$ (see Fig.. 10). After further application of f each of these rectangles is mapped into another one. Therefore, we can conclude that the set

$$\Lambda = \left(\cup_{n \in \mathbb{N}} f^{-n}(H_0 \cup H_1) \right) \cup \left(\cup_{n \in \mathbb{N}} f^n(V_0 \cup V_1) \right) \tag{40}$$

is an invariant set for the map f. By construction this set consists of infinitely many points lying at the intersection of horizontal and vertical segments, has measure zero and each point of Λ is the limiting point of it. Therefore, Λ is a Cantor set.

Moreover, we can describe the dynamics of f on Λ by constructing the homeomorphism of the latter into the phase space Σ of *symbolic dynam-*

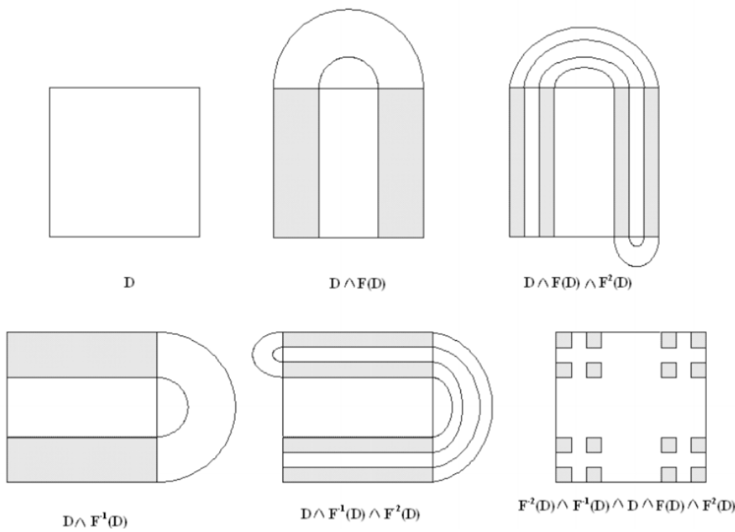

Figure 10: Iterative construction of $f^{-2}(S) \cap f^{-1}(S) \cap S \cap f^1(S) \cap f^2(S)$.

ics considered in paragraph 5.1. Indeed, for each $x \in \Lambda$ let us define the corresponding bi-infinite sequence $s \in \Sigma$ as

$$s = h(x) = \{s_i\}_{i=-\infty}^{+\infty}, \text{ where } f^i(x) \in H_{s_i},\ s_i \in \{1, 2\}. \tag{41}$$

One can show that thus defined map $h : \Lambda \to \Sigma$ is homeomorphism with respect to the standard metric of $S \subset \mathbb{R}^2$ and the distance (1) of Σ. To verify that h is surjective, take a sequence $s \in \Sigma$ and for each $n \in \mathbb{N}$ consider the set $R_n(s)$ of all points $x \in S$, not necessarily belonging to Λ, such that

$$f^k(x) \in H_{s_k} \text{ for all } -n \leq k \leq 0 \text{ and } f^k(x) \in V_{s_k} \text{ for all } k > 0.$$

For example, for $n = 1$ the set $R_1(s)$ is one of the four intersections $V_i \cap H_j$. In general, $R_n(s)$ belongs to the intersection of a vertical and a horizontal stripes. These stripes are getting thinner and thinner as $n \to \infty$, approaching in the limit a vertical and a horizontal segment, respectively. Such segments obviously intersect at a single point $x \in S$ with $h(x) = s$. Thus, $h : \Lambda \to \Sigma$ is a one-to-one map. This implies, in particularly, that Λ is uncountable.

Finally, we observe that Smale's horseshoe map acting in Λ is topologically equivalent to the *Bernoulli shift* map in Σ, i.e. it holds

$$f(x) = h^{-1}(\sigma(h(x)) \text{ for all } x \in \Lambda.$$

Indeed, if $y = f(x) \in \Lambda$ then the corresponding bi-infinite sequence $s' = h(y) \in \Sigma$ is obtained by a single shift of all indices of $s = h(x)$ to the left, i.e. $s' = \sigma(s)$, where we used definition (2). The topological equivalence between f and σ is summarised in the diagram in Fig. 11 This combined

$$\begin{tikzcd} \Lambda \arrow[r, "f"] \arrow[d, "h"'] & \Lambda \arrow[d, "h"] \\ \Sigma \arrow[r, "\sigma"] & \Sigma \end{tikzcd}$$

Figure 11: Topological equivalence between Smale's horseshoe and Bernoulli shift maps.

together with Theorem 1 and Definition 5.1 implies that Λ is the *chaotic set* for Smale's horseshoe map with the following properties.

Theorem 5. *The horseshoe map f has a closed invariant set Λ that contains a countable set of periodic orbits of arbitrarily long period, and an uncountable set of non-periodic orbits, among which there is an orbit passing arbitrarily close to any point of Λ. Moreover, f is structurally stable. Namely, for all sufficiently small ε, $||f - f_1||_{C^1(S)} \leq \varepsilon$ implies that the map f_1 has an invariant Cantor set Λ_1, such that the dynamical system $f_1|_{\Lambda_1}$ is topologically equivalent to $f|_\Lambda$.*

Remark 5.4. The last statement on the structural stability of f restricted to set Λ follows from the following observation. If Jacobians of f_1 in H_0 and H_1 are close to constant ones of f (defined through (38) and (39), respectively) then the images of these two horizontal stripes are still disjoint and close to two vertical stripes V_0 and V_1, respectively. This implies that qualitatively the dynamics of f_1 is the same as of f. Namely, the set $\cup_{i=0}^{n} f_1^i(S)$ will be consisting of 2^n vertical thin disjoint curve-linear stripes and $\cup_{i=-n}^{0} f_1^i(S)$ of 2^n horizontal ones. Therefore, the set Λ_1 of all points x, such that $f_1^n(x) \in S$ for all $n \in \mathbb{Z}$ is again homeomorphic to Σ and the dynamics of f_1 in it is equivalent to the one of shift map σ.