Consultation questions

B5.6 Nonlinear Systems
Prof. Alain Goriely

Do not turn this page until you are told that you may do so

1. [25 marks] Consider the system

$$\dot{x} = x(3 - x - 5y),$$
$$\dot{y} = y(-1 + x + y).$$

with $(x, y) \in \mathbb{R}^2$.

(a) [5 marks] Show that both axes are invariant sets and that the line $J = x + 3y - 3 = 0$ is also an invariant set. Show that any intersection of these invariant sets defines a fixed point.

(b) [5 marks] Use linear analysis to determine the stability of the origin and show that there is a non-hyperbolic fixed point at $(1/2, 1/2)$.

(c) [5 marks] Find a such that $H = xy^3 J^a$ is a first integral of the system ($\dot{H} = 0$).

(d) [5 marks] Use the first integral H to determine the stability of the non-hyperbolic fixed point.

(e) [5 marks] Given that the three fixed points computed in (a) are unstable and hyperbolic, sketch the phase portrait of the system.

SOLUTION

(a) Since $\dot{x} = 0$ when $x = 0$ and $\dot{y} = 0$ when $y = 0$, both axes are invariant. Similarly $\dot{J} = J(y - x)$ and $\dot{J} = 0$ when $J = 0$.

The intersection of these three invariant sets define three points:

$$\{\{x \to 0, y \to 1\}, \{x \to 3, y \to 0\}, \{x \to 0, y \to 0\}\},$$

and a direct computation shows that they are indeed fixed.

(b) The system has four fixed points:

$$\left\{\{x \to 0, y \to 1\}, \{x \to 3, y \to 0\}, \{x \to 0, y \to 0\}, \left\{x \to \frac{1}{2}, y \to \frac{1}{2}\right\}\right\}.$$

The origin is unstable with trivial eigenvalue $\{3, -1\}$. The last one has eigenvalues $\pm \mathrm{i}$ and is therefore non-hyperbolic.

(c) After simplification $\dot{H} = (-2+a)xy^3(-x+y)(-3+x+3y)^a$. Hence $\dot{H} = 0$ when $a = 2$.

(d)The function $V = 1/16 - H$ vanishes at (1/2,1/2), is positive close to (1/2,1/2) and satisfies all the conditions to be a Lyapunov function for the point $x = y = 1/2$. Hence $(1/2, 1/2)$ is stable.

(e)The direction of the arrows is obtained from the knowledge of the unstable and stable direction of the origin.

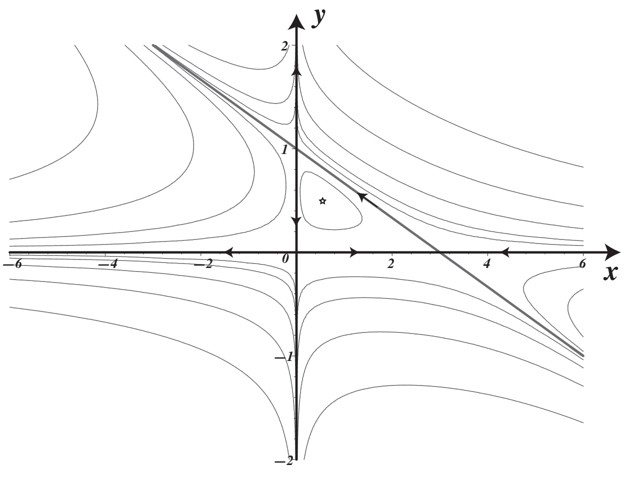

Figure 1: Phase portrait of the system.

2. [25 marks] Consider the tent map, mapping the unit interval $[0, 1]$ into itself and defined by

$$x_{n+1} = \begin{cases} 2x_n & 0 \leqslant x_n \leqslant 1/2, \\ 2(1 - x_n) & 1/2 \leqslant x_n \leqslant 1 \end{cases} \tag{1}$$

(a) [5 marks] Sketch the tent map and prove that it defines a map of the unit interval into itself.

(b) [5 marks] Find all fixed points and determine their stability.

(c) [5 marks] Show that there is a single period-2 orbit and analyse its stability.

(d) [5 marks] Show that any initial rational value less than one ends up on a periodic orbit and analyse the stability of this periodic orbit.

(e) [5 marks] The Lyapunov exponent for a map $x_{n+1} = f(x_n)$ is defined as

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} |f'(x_k)|. \tag{2}$$

Compute the Lyapunov exponent for the tent map and show that it is positive (hence, the system is chaotic).

SOLUTION

(a) If $0 \leqslant x_n \leqslant 1/2$, then $0 \leqslant x_{n+1} \leqslant 1$. Similarly, if $1/2 \leqslant x_n \leqslant 1$, then $0 \leqslant x_{n+1} \leqslant 1$.

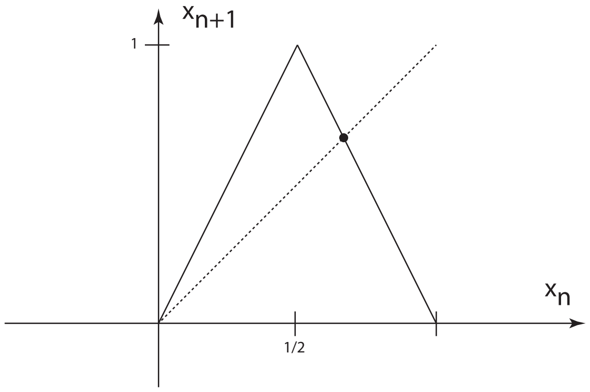

Figure 2: The tent map.

(b) From the sketch, we see that there are 2 fixed point: $x = 0$ and a second one located in the second half of the interval. So, we have $x_* = 2(1 - x_*)$, that is $x_* = 2/3$. Since we have $|f'(2/3)| = 2 > 1$, the fixed point is unstable. The fixed point at 0 is also unstable since $f'(0) = 2 > 1$.

(c) To find the period-2 orbit, we can start x_0 either on the left (L) interval $[0, 1/2]$ or on the right (R) interval $[1/2, 1)$. Similarly, the first iterate can be on L or R. We consider the four different cases:

LL: If x_0 and x_1 are in L, we have $x_2 = 2(x_1) = 2(2x_0)$ so that $x_2 = x_0$ if $x_0 = 0$ but that is a fixed point.

RR: We have a period-2 orbit if $x_0 = 2(1 - (2(1 - x_0)))$. That is, $x_0 = 2/3$. Again, this is just the fixed point.

LR: We have a period-2 orbit if $x_0 = 2(1 - (2x_0))$, that is $x_0 = 2/5$.

RL: We have a period-2 orbit if $x_0 = 2(2(1 - x_0))$, that is $x_0 = 4/5$.

We conclude that there is a single periodic orbit connecting 2/5 to 4/5.

The stability of a periodic orbit is given by $f'(x_0)f'(x_1) = -4 < -1$. So the orbit is unstable.

(d) Consider orbits starting at a rational number $x_0 = p/q$ where $p < q$. The map sends this initial value to another rational number of the form p'/q. Since there are at most $(q - 1)$ numbers of the form p/q, the orbit of x_0 eventually repeats this number and the period is at most q. Let $k \leqslant q$ be the period of the orbit. Then, for a certain n its stability is given by $|\prod_{i=n+1}^{n+k} f'(x_i)| = 2^k > 1$ and we conclude that all periodic orbits are unstable. Note that $x = 1/2$ is never part of a periodic orbit as $x_0 = 1/2$ leads to $x_2 = 0$, a fixed point. Therefore $f'(x_i)$ is always well defined.

(e) Since $|f'(x_n)| = 2$ for all $x_n \neq 1/2$, we have $\lambda = 2 > 0$ and the system is chaotic.

3. [25 marks] Consider the map, from the unit interval $[0, 1]$ into itself, defined by

$$x_{n+1} = f(x_n), \tag{3}$$

for $n = 0, 1, 2, \dots,$ where x_0 is given.

(a) [5 marks] For a general $f(x)$, what is a *fixed point*? Explain how to determine its stability through linear analysis. Extend these results to periodic orbits by giving conditions that guarantee the existence of stable orbits of period p (where $p > 1$ is an integer).

(b) [7 marks] The map (3) is *unimodal* if $f(x)$ is continuous and there exists a point $a \in (0, 1)$ such that f is strictly increasing in $[0, a)$ and strictly decreasing on $(a, 1]$. Show that if $f(a) \leqslant a$ then all solutions tend to fixed points which lie in $[0, f(a)]$.

(c) [13 marks]

(i) Consider the particular function $f(x) = \mu \sin(\pi x)$, where μ is a parameter in $[0, 1]$. Show that the point $x = 0$ is a fixed point. Determine its stability, and determine the value μ^* at which it loses its stability. Write the normal form at the bifurcation and state the type of bifurcation.

(ii) Determine the stability of the new bifurcating fixed point in a neighbourhood of μ^*. Show that this fixed point is stable for values of μ such that $\mu^* < \mu < 1/2$.

SOLUTION

(a) A fixed point x_0 is such that $f(x_0) = x_0$. It is stable when $|\lambda| < 1$, where $\lambda = Df(x_0)$. An orbit of period p exists if there exists x_0 such that $f^{(p)}(x_0) = x_0$ and $x_k = f^{(k)}(x_0) \neq x_0$ for $k = 1, \ldots, p-1$. It is stable if $|\prod_{i=1}^{p} f'(x_i)| < 1$.

(b) let F be the map associated with f. Then $F([0,a]) = F([a,1]) = [0, F(1)] \subseteq [0,a]$. Starting at x_0, after one iteration $x_1 \in [0,a]$. In that interval $x < y$ implies $f(x) < f(y)$. Therefore: If $x_1 < f(x_1)$ then x_i increases monotonically to the nearest fixed point. If $x_1 > f(x_1)$ then x_i decreases monotonically to the nearest fixed point.

(c) The fixed point $x = 0$ is stable until $\mu\pi = 1$, that is $\mu < 1/\pi$. The normal form is obtained by expanding $f(x,\mu)$ around $x = 0$ and $\mu = 1/\pi$. Defining $x = \sqrt{6}/\pi y$ and $\mu = 1/\pi(1+\lambda)$ we obtain

$$y_{n+1} = (1+\lambda)(y_n - y_n^3) + O(5). \tag{4}$$

This is the normal form of a supercritical pitchfork bifurcation.
A direct analysis of the normal form reveals that in a neighbourhood of μ^*, the new fixed point is stable (with multiplier $1 - 2\lambda$, hence stable for λ small). Using the general result about unimodal map and the fact that there are only two fixed points (one of which is unstable), we conclude that the new fixed point is stable until $f(1/2) = 1/2$ that is until $\mu = 1/2$.

4. Consider the two-dimensional system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mu + x^2 + xy + y^2,$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2\mu - y + x^2 + xy,$$

where μ is a constant parameter.

(a) [10 marks] First, fix $\mu = 0$.

(i) Find a quadratic approximation of the centre manifold in a neighbourhood of the origin $(x, y) = (0, 0)$.

(ii) Find a quadratic vector field that approximates the dynamics on the centre manifold, in a neighbourhood of the origin. Use this vector field to determine the stability of the origin. Sketch the phase portrait in a neighbourhood of the origin.

(b) [15 marks] Now, consider the case $\mu \neq 0$ and the extended three-dimensional system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mu + x^2 + xy + y^2,$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2\mu - y + x^2 + xy,$$
$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = 0.$$

(i) Find a quadratic approximation of the centre manifold in a neighbourhood of the origin $(x, y, \mu) = (0, 0, 0)$.

(ii) Find a quadratic vector field that approximates the dynamics on the centre manifold in a neighbourhood of the origin $(0, 0, 0)$, Determine the type of bifurcation occurring at the origin for the original two-dimensional system when $|\mu| \ll 1$.

SOLUTION

(a) [10 marks] The linear centre space is along the x-axis so we write $y = h(x) = ax^2 + 0(3)$. Therefore

$$\dot{y} = 2ax\dot{x} = 2ax(x^2 + xy + y^2) = 2ax(x^2 + xax^2) + O(4)$$

This is equal to

$$\dot{y} = -ax^2 + x^2 + xax^2 + O(4)$$

Hence, we conclude that to lowest order $a = 1$ and $y = x^2 + O(3)$ is the quadratic approximation of the centre manifold. Replacing $y = x^2 + O(3)$ in the first equation gives $\dot{x} = x^2 + O(3)$ and we conclude that $x = 0$ is unstable. The phase portrait is given in Figure 3.

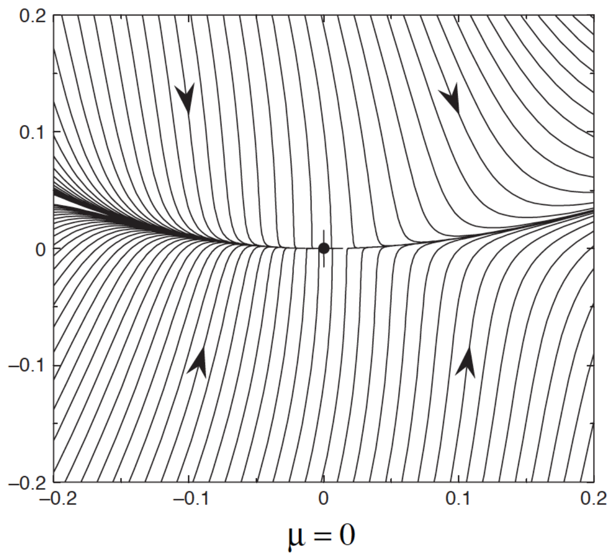

Figure 3: Phase portrait of the system for $\mu = 0$.

(b) [15 marks] Following the same steps, we find that for the extended system, the centre manifold is

$$y = 2\mu + x^2 + O(3).$$

The dynamics on the centre manifold is

$$\dot{x} = \mu + x^2 + 2\mu x + 4\mu^2 + O(3).$$

For $|\mu| \ll 1$, there is a saddle-node bifurcation at $\mu = 0$ with two fixed points when $\mu = 0$ and no fixed point when $\mu > 0$. For $\mu < 0$, there are two fixed points: an unstable saddle at $(x, y) = (\sqrt{\mu}, 0) + O(\mu)$ and a stable node at $(x, y) = (-\sqrt{\mu}, 0) + O(\mu)$. The phase portrait is given in Figure 4.

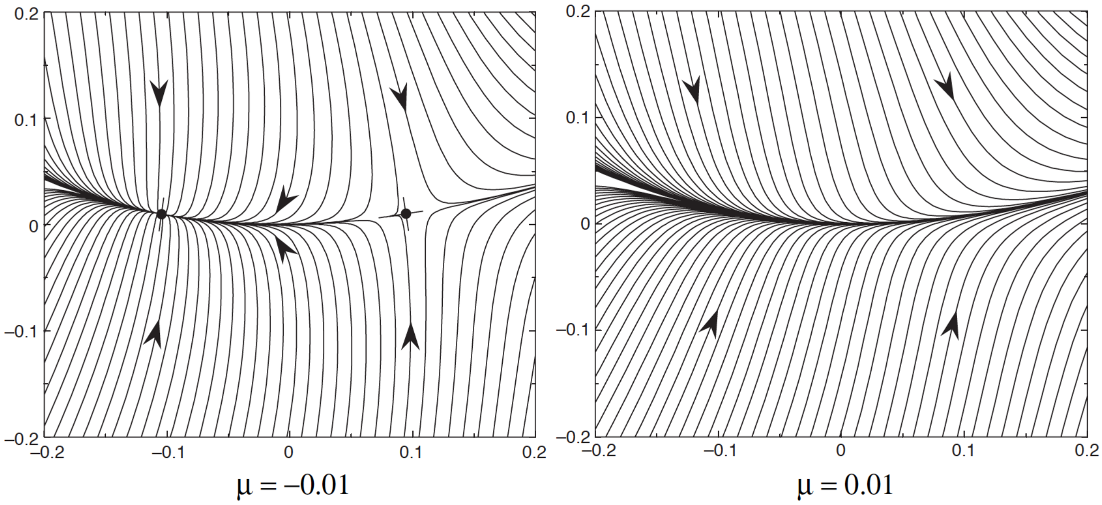

Figure 4: Phase portrait of the system for $\mu \neq 0$.

5. Consider the forced system

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= x - x^2, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -y + 2xy + \epsilon \cos t,\end{aligned} \tag{5}$$

where $0 < \epsilon \ll 1$ is a parameter.

(a) [5 marks] Define a *first integral* for a general n-dimensional system of the form $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. Show that for $\epsilon = 0$, the two-dimensional system (5) supports a first integral of the form $J = y(ax + bx^2)$. Determine a and b.

(b) [5 marks] Define a *hyperbolic fixed point* for a general n-dimensional system. Consider the two-dimensional system (5) in the case $\epsilon = 0$. Find the fixed points and determine their stability. Which fixed point is hyperbolic? Find and sketch all invariant sets that contain at least one fixed point.

(c) [5 marks] Define *homoclinic* and *heteroclinic* orbits for a general n-dimensional system. Show that for $\epsilon = 0$, system (5) has a heteroclinic orbit of the form

$$(x(t), y(t)) = (\frac{1}{1 + e^{ct}}, 0),$$

and determine the constant c.

(d) [5 marks] Define a *conservative system* for a general n-dimensional system, and explain how to test if a system is conservative. Show that the two-dimensional system (5) is conservative for $\epsilon = 0$.

(e) [5 marks] Define the *Melnikov integral* for a general two-dimensional system of the form $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \epsilon \mathbf{g}(\mathbf{x}, t)$ when the unforced system is conservative. What is the geometric interpretation of the vanishing of the Melnikov integral? For the two-dimensional system (5), compute the Melnikov integral.

[*Hint: you can use without proof that*

$$\int_{-\infty}^{+\infty} \frac{\cos t}{\cosh^2(\frac{t}{2})}\,\mathrm{d}t = 4\pi \mathrm{csch}(\pi), \qquad \int_{-\infty}^{+\infty} \frac{\sin t}{\cosh^2(\frac{t}{2})}\,\mathrm{d}t = 0. \qquad]$$

SOLUTION

Note for each question there is a theoretical part (worth about 2 points) and a practical part for about 3 points. The computations are not difficult but answering all these questions and the theory attached to it would demonstrate a very good understanding of the material.

(a) [5 marks] A first integral $J = J(\mathbf{x})$ is a scalar function such that is constant on all trajectories, that is $\dot{J} = \nabla J \cdot \mathbf{f} = 0$. A direct computation gives $a = -b$, so we can choose $a = 1 = -b$.

(b) [5 marks] A fixed point is hyperbolic if all the eigenvalues of its Jacobian matrix have non-zero real parts. For our problem the fixed points are $(0, 0)$ and $(1, 0)$. The Jacobian matrix is

$$D\mathbf{f} = \begin{pmatrix} 1-2x & 0 \\ 2y & 2x-1 \end{pmatrix}$$

with eigenvalues $(1, -1)$ and $(-1, 1)$, respectively. Both fixed points are saddle node. They are unstable and hyperbolic. There are nine invariant sets that cannot be further decomposed: the fixed points, the positive and negative y-axis, the negative x-axis, the interval between $x = 0$ and $x = 1$ on the y-axis, the interval between $x = 0$ and $x = +\infty$ on the y-axis, the two vertical lines defined by $x = 1$ and y either positive or negative. In addition, the unions of any of these sets is also invariant.

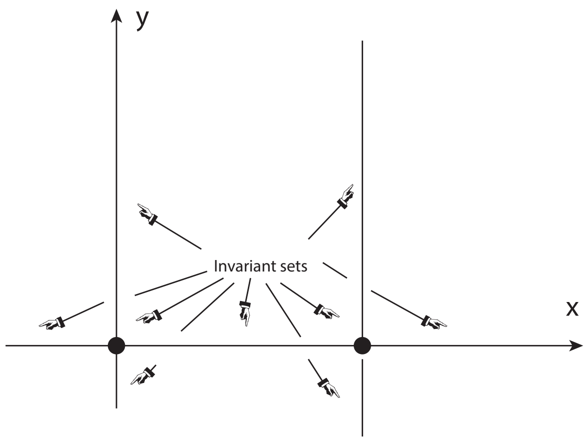

Figure 5: The invariant sets of problem 3b.

(c) [5 marks] A Homoclinic orbit connects a fixed point to itself and a heteroclinic orbit connects two different fixed points. The system has a heteroclinic orbit between the two fixed point along the x axis and a direct substitution leads to $c = -1$.

(d) [5 marks] A conservative system is such that the dynamics conserves volume elements in phase space. Through, Liouville's theorem, a direct test is that $\mathrm{Tr}(\mathrm{D}\mathbf{f}(\mathbf{x}(\mathbf{t}))$ vanishes identically. A direct computation shows that $\mathrm{Tr}J = 0$. Hence the system is conservative.

(e) [5 marks] For a two-dimensional system of the form $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \epsilon \mathbf{g}(x, t)$, The Melnikov integral is given by

$$M(t_0) = \int_{-\infty}^{+\infty} \mathbf{f}(\hat{\mathbf{x}}(t)) \wedge \mathbf{g}(\hat{\mathbf{x}}(t), t + t_0) \, \mathrm{d}t, \tag{6}$$

where $\mathbf{f} \wedge \mathbf{g} = (f_1 g_2 - f_2 g_1)$ and $\hat{\mathbf{x}}(t)$ is a homoclinic or heteroclinic orbit. The Melnikov integral gives a measure of the distance between stable and unstable manifolds under perturbation. The vanishing of the integral is associated with either the preservation of the orbit or a transverse intersection, leading in the homoclinic case to the existence of a Smale horseshoe for the dynamics. After simplification, and using the hint, the Melnikov integral for the system is

$$M(t_0) = \pi \text{csch}(\pi) \cos(t_0). \tag{7}$$

However, the perturbed dynamics does not have a Smale horseshoe since the stable and unstable manifolds of the fixed points extend to infinity and the system does not exhibit chaos (but the students have not seen that and any conclusion regarding the existence of a chaotic set will be disregarded).