Question Sheet 3

The first two questions are optional. They reiterate material covered in lectures, but are well worth doing yourself.

(i) [*Optional*]

(a) Use contour integration to show that

$$\int_{-R}^{R} \mathrm{e}^{\pm \mathrm{i}s^2}\,\mathrm{d}s \to (1 \pm \mathrm{i})\sqrt{\frac{\pi}{2}} + \mathrm{O}\left(\frac{1}{R}\right) \quad \text{as } R \to \infty.$$

(b) If $f(k)$ is continuously differentiable, show that

$$\int_a^b f(k)\mathrm{e}^{\mathrm{i}kt}\,\mathrm{d}k = \mathrm{O}(1/t) \quad \text{as} \quad t \to \infty.$$

(c) Let

$$I(t) = \int_a^b f(k)\mathrm{e}^{\mathrm{i}\psi(k)t}\,\mathrm{d}k,$$

where $f(k)$ is continuously differentiable, $\psi(k)$ is real-valued and twice continuously differentiable on $[a, b]$, and suppose that $\psi'(k)$ has a single simple zero at $k = k^* \in (a, b)$. Explain schematically why you would expect the behaviour of $I(t)$ as $t \to \infty$ to be dominated by values of k where $\psi(k)$ is stationary.

By splitting the range of integration, and performing appropriate changes of variables in the integrals, show that

$$I(t) \sim f(k_*)\mathrm{e}^{\mathrm{i}\left(\psi(k_*)t \pm \pi/4\right)}\sqrt{\frac{2\pi}{|\psi''(k_*)|\,t}} \quad \text{as } t \to \infty,$$

where the $\pm$ takes the sign of $\psi''(k_*)$.

(ii) [*Optional*] Show that the Fourier transform of $\epsilon/(x^2 + \epsilon^2)$ is $\pi\mathrm{e}^{-\epsilon|k|}$.

If fluid occupying the half-space $z < 0$ starts from rest with the initial free surface profile $\eta_0(x) = -a\epsilon/\pi(x^2 + \epsilon^2)$, show that

$$\eta(x, t) = -\frac{a}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\epsilon|k|}\cos\left(t\sqrt{g|k|}\right)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k.$$

Invoke the method of stationary phase to show that, as $x, t \to \infty$, the major contribution to the integral comes from values of k satisfying

$$\frac{x}{t} \pm \frac{\sqrt{g|k|}}{2k} = 0.$$

Hence show that, for $x > 0$, and if ϵ is sufficiently small,

$$\eta \sim -\frac{at}{2}\sqrt{\frac{g}{\pi x^3}}\cos\left(\frac{gt^2}{4x} - \frac{\pi}{4}\right).$$

Sketch this approximation of $\eta(x, t)$ versus x for a fixed (large) value of t.

1. Steady small-amplitude waves disturb a fluid of constant density ρ occupying the half-space $z < 0$ and flowing with uniform velocity $U\hat{\boldsymbol{e}}_x$. Show that the disturbance velocity potential $\phi(x, y, z)$ and free surface displacement $z = \eta(x, y)$ satisfy

$$\begin{aligned} \nabla^2\phi &= 0 && z < 0, \\ \frac{\partial\phi}{\partial z} = U\frac{\partial\eta}{\partial x}, \ U\frac{\partial\phi}{\partial x} + g\eta &= 0 && z = 0, \\ \boldsymbol{\nabla}\phi &\to \boldsymbol{0} && z \to -\infty. \end{aligned}$$

Show that separable solutions with $\eta = A\mathrm{e}^{\mathrm{i}(kx+\ell y)}$ exist provided

$$k^2 = \frac{g^2}{2U^4}\left(1 + \sqrt{1 + \frac{4U^4\ell^2}{g^2}}\right). \qquad (\star)$$

An obstacle is placed on the y-axis such that $\eta = \eta_0(y)$ and $\partial\eta/\partial x = 0$ at $x = 0$. Deduce that

$$\eta(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{\eta_0}(\ell)\mathrm{e}^{\mathrm{i}\ell y}\cos\big(k(\ell)x\big)\,\mathrm{d}\ell,$$

in $x > 0$, where $k(\ell)$ is given by $(\star)$ and $\widehat{\eta_0}$ is the Fourier transform of η_0.

If $y = \lambda x$, where x is large, show that the main contribution to η arises from values of ℓ such that $\lambda = \pm\mathrm{d}k/\mathrm{d}\ell$. Hence show that the far-field wave pattern is contained in the region

$$|\lambda| = \sqrt{\frac{s-1}{2s^2}} \leqslant \frac{1}{2\sqrt{2}},$$

where $s = \sqrt{1 + 4U^4\ell^2/g^2}$.

2. A barotropic compressible inviscid fluid flows steadily at uniform speed U in the x-direction past a thin symmetric wing with upper and lower surfaces given by $y = \pm f(x)$, $-a < x < a$.

   (a) If the velocity is irrotational and given by $\boldsymbol{u} = U\hat{\boldsymbol{e}}_x + \boldsymbol{u}'$, where $|\boldsymbol{u}'| \ll U$, show that $\boldsymbol{u}' = \boldsymbol{\nabla}\phi$ for some velocity potential ϕ. Find an expression for the pressure in terms of ϕ and show that two-dimensional steady disturbances are governed by the equation

$$\left(1 - M^2\right)\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0,$$

   where $M = U/c_0$ and c_0 is the speed of sound.

   (b) Assuming that the flow is symmetric about the x-axis, find the boundary conditions for ϕ linearised to $y = 0$. Deduce that in subsonic flow the Fourier transform of ϕ with respect to x satisfies

$$\frac{\partial^2\widehat{\phi}}{\partial y^2} = \beta^2k^2\widehat{\phi} \quad y > 0, \qquad\qquad \frac{\partial\widehat{\phi}}{\partial y} = U\widehat{F}(k) \quad y = 0,$$

   where $\beta = \sqrt{1 - M^2}$ and $\widehat{F}(k)$ is the Fourier transform of

$$F(x) = \begin{cases} f'(x) & |x| < a, \\ 0 & |x| > a. \end{cases}$$

(c) Hence find $\partial\phi/\partial y$ and deduce that

$$\phi(x,y) = \frac{U}{2\pi\beta}\int_{-a}^{a} f'(\xi)\log\left((x-\xi)^2+\beta^2y^2\right)\,\mathrm{d}\xi.$$

[*You may use without proof the result that* $\pi\mathrm{e}^{-\epsilon|k|}$ *is the Fourier transform of* $\epsilon/(x^2+\epsilon^2)$ for $\epsilon > 0$.]

(d) Show that this problem is mathematically equivalent to *incompressible* flow past the same body and deduce that the overall force on the wing in subsonic flow is zero.

3. Show that for homentropic flow, the equations of one-dimensional gas dynamics

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho u) = 0, \qquad \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

can be written as

$$\left(\frac{\partial}{\partial t}+(u\pm c)\frac{\partial}{\partial x}\right)\left(u\pm\frac{2c}{\gamma-1}\right) = 0,$$

where $c^2 = \gamma p/\rho$.

4. Gas is confined to a semi-infinite tube $x > s(t)$ by a piston at $x = s(t)$. At $t = 0$ the gas is at rest, with $s = 0$, $p = p_0$ and $\rho = \rho_0$.

(a) If the piston expands the gas such that $\dot{s}(0) = 0$ and $\dot{s}(t), \ddot{s}(t) < 0$ for $t > 0$, show that the flow is given parametrically by

$$u = \dot{s}(\tau), \qquad u - \frac{2c}{\gamma-1} = -\frac{2c_0}{\gamma-1}, \qquad x = s(\tau)+(u+c)(t-\tau),$$

in the region $x < c_0 t$, where $c_0^2 = \gamma p_0/\rho_0$. Show that the piston leaves the gas behind if $\dot{s}(t) < -2c_0/(\gamma-1)$ for any t.

(b) If the piston is removed instantaneously, show that $u + c = x/t$ and deduce that the gas expands into the region

$$-\frac{2c_0}{\gamma-1} < \frac{x}{t} < c_0.$$

(c) If $s(t) = -\alpha t^2$, where $\alpha > 0$, show that

$$u(x,t) = \frac{1}{\gamma}\left\{\sqrt{\left(c_0+(\gamma+1)\alpha t\right)^2+4\gamma\alpha(x-c_0t)} - c_0 - (\gamma+1)\alpha t\right\} \qquad (\star)$$

in $x < c_0 t$, and find where the vacuum forms.

(d) If $\alpha < 0$, i.e. the piston is pushed in to the gas, show that $(\star)$ still applies for $t < -c_0/(\gamma+1)\alpha$. What happens when $t > -c_0/(\gamma+1)\alpha$?