

## Problem Set 1

For this sheet assume the Empty Set Axiom, the Axioms of Extensionality, Pairs, Unions and the Comprehension Scheme. In question 1 **only** assume also the Powerset Axiom: if  $X$  is a set there is a set  $\mathcal{P}(X)$  whose elements are precisely the subsets of  $X$  (this set is called the powerset of  $X$ ).

1. For each statement give either a proof or a counterexample. (Where an intersection  $\bigcap c$  arises assume the set  $c$  is non-empty. You may assume that all the sets involved exist.)

- (i)  $a \setminus \bigcap b = \bigcup \{a \setminus x : x \in b\}$
- (ii)  $(\bigcap a) \cup (\bigcap b) = \bigcap \{x \cup y : x \in a, y \in b\}$
- (iii)  $\mathcal{P}(\bigcup X) = X$
- (iv)  $\bigcup \mathcal{P}(X) = X$
- (v) If  $\mathcal{P}(a) \subseteq \mathcal{P}(b)$  then  $a \subseteq b$ .

2. (a) Prove that the unordered pair  $\{x, y\}$  of  $x$  and  $y$  is the unique set whose elements are precisely  $x$  and  $y$ .

(b) Let  $\phi(z, w_1, \dots, w_k)$  be a formula of  $\mathcal{L}$  and  $w_1, \dots, w_k, x$  sets. Prove that the subset  $y$  of  $x$  afforded by the Comprehension Scheme is unique with the stated property.

3. Let  $a$  be a set. Prove that  $\{a\} \times \{a\} = \{\{\{a\}\}\}$ .

4. (a) Show that if we define an ordered triple  $(a, b, c)$  of sets to be  $\langle\langle a, b \rangle, c\rangle$  then this definition “works”: i.e. if  $(a, b, c) = (a', b', c')$  then  $a = a', b = b', c = c'$ . You may use the fact (from lectures) that  $\langle a, b \rangle$  “works”.

(b) for each of the following alternative possible definitions of an ordered triple, prove that the definition “works” or give a counterexample.

- (i)  $(a, b, c)_1 = \{\{a\}, \{a, b\}, \{a, b, c\}\}$
- (ii)  $(a, b, c)_2 = \{\langle 0, a \rangle, \langle 1, b \rangle, \langle 2, c \rangle\}$  (where  $0 = \emptyset, 1 = \{0\}, 2 = \{0, 1\}$ )
- (iii)  $(a, b, c)_3 = (\{0, a\}, \{1, b\}, \{2, c\})$  where  $(., ., .)$  is as in part (a)
- (iv)  $(a, b, c)_4 = \{\{0, a\}, \{1, b\}, \{2, c\}\}$ .

5. A set  $a$  is called *transitive* if, for all sets  $x$ , if  $x \in a$  then  $x \subseteq a$ . I.e. if  $\bigcup a \subseteq a$ . Prove that

- (i)  $\emptyset$  is transitive
- (ii) if  $a$  is transitive then so is  $a \cup \{a\}$  (this set is denoted  $a^+$ )
- (iii)  $a$  is transitive iff  $\bigcup (a \cup \{a\}) = a$
- (iv)  $a$  is transitive iff, for all sets  $x, y$ , if  $x \in y \in a$  then  $x \in a$
- (v) the intersection of any (non-empty) set of transitive sets is transitive
- (vi) the union of any set of transitive sets is transitive
- (vii) write a formula in  $\mathcal{L}$  with a free variable  $x$  expressing “ $x$  is transitive”.

6. Prove that

- (i) if  $x$  is a set, there is no set containing every set that is not an element of  $x$ .
- (ii) there is no set of all one-element sets.
- (iii) there is no set of all two-element sets.

7. (a) Prove that

- (i) if  $a, b, c$  are sets then  $\{a, b, c\}$  is a set.
- (ii) if  $x_1, \dots, x_n$  are sets then  $\{x_1, \dots, x_n\}$  is a set (here  $n \in \mathbb{N}$ ).
- (iii) if  $X$  is a finite set then  $\mathcal{P}(X)$  is a set (do not assume the Powerset Axiom!).
- (iv) if  $X$  is a finite set then the collection of all two-element subsets of  $X$  is a set.

(b) Suppose  $X$  is a set all of whose elements are finite sets. Prove that there is a set  $Y$  consisting of all the elements of  $X$  that have an *even* number of elements. (Note it is not sufficient that  $Y$  “is” a subset of  $X$ .)

Hint: You must not use the power set axiom. However, from part (a) you know that if  $X$  is a finite set then there is a set which is its powerset.

8. Prove that there exist infinitely many sets.