Problem Set 2

1. (a) Let R be a relation (that is R is a set of ordered pairs). Prove that $\mathrm{dom}(R)$, which we define to be $\{x : \exists y\big((x, y) \in R\big)\}$, is a set.

(b) Let X, Y be sets. Prove there exists a set whose elements are the functions from X onto Y. I.e. surjections.

(c) Let X be a set. Prove that there is a set consisting precisely of all strict total orders on X.

2. Prove, using induction and the fact that each $n \in \omega$ is transitive (by sheet 1) that $n \in n$ is false for every $n \in \omega$. (Do not use the Axiom of Foundation.)

3. Prove that the function sending n to $n!$ exists (as a set). Hint: Use Recursion in the form of Theorem 5.1 with $X = \omega \times \omega$.

4. Prove that multiplication on ω is commutative by proving the following statements for $n, m \in \omega$ by induction. You may use the other arithmetic properties established in lectures; or (hint) prove that $m * n = n.m$ satisfies the same recursion as $m.n$.

(i) $0.n = 0$
(ii) $m^+.n = m.n + n$
(iii) $m.n = n.m$

5. Write $1 = 0^+, 2 = 1^+$. Define $n \in \omega$ to be *even* if it is of the form $2.k$ for some $k \in \omega$ and *odd* if it is of the form $2.h + 1$ for some $h \in \omega$. Prove that

(i) every element of ω is either even or odd
(ii) no element of ω is both even and odd

6. A *Peano system* is a triple (A, s, a_0) in which A is a set, $a_0 \in A$, and $s : A \to A$ is a function with is (a) one-to-one, (b) does not include a_0 in its range, and (c) satisfies the Principle of Induction: that is, if $S \subseteq A$, $a_0 \in S$ and $\forall a(a \in S \to s(a) \in S)$, then $S = A$.

(i) Prove that $(\omega, x \mapsto x^+, 0)$ is a Peano system.

(ii) Suppose (A, s, a_0) is a Peano system. Prove that there exists an isomorphism from $(\omega, ^+, 0)$ to (A, s, a_0), that is, there is a bijection $f : \omega \to A$ such that $f(0) = a_0$ and, for all $n \in \omega$, $f(n^+) = s(f(n))$.

[*Hence, up to isomorphism,* $(\omega, ^+, 0)$ *is the unique Peano system.*]

Hint: Define f by recursion and verify the required properties.

PTO

7. Let $X = X_0$ be a set. By the axiom of Unions, the sets $X_1 = \bigcup X, X_2 = \bigcup X_1, \ldots$ are sets. The *transitive closure* of X is defined to be $T(X) = \bigcup_{n=0}^{\infty} X_n = \bigcup\{X_0, X_1, \ldots\}$. Prove that

(i) $T(X)$ is a set
(ii) $T(X)$ is transitive
(iii) $X \subseteq T(X)$
(iv) If $X \subseteq Y$ and Y is transitive than $T(X) \subseteq Y$
(v) If X is transitive then $T(X) = X$.

8. A set X is called *hereditarily finite* if its transitive closure $T(X)$ is a finite set.

(i) Prove that the following sets are hereditarily finite

$$\emptyset, \quad \{\emptyset\}, \quad \{\emptyset, \{\emptyset\}\}, \quad \{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$$

(ii) Prove that a subset of a hereditarily finite set is hereditarily finite, and an element of a hereditarily finite set is hereditarily finite. [You may assume: a subset of a finite set is finite]

(iii) Let $\mathbf{H}$ be the class of hereditarily finite sets. It turns out that $\mathbf{H}$ is in fact a set. Prove that the Empty Set Axiom, the Axioms of Extensionality, Pairs, Unions and the Comprehension Scheme are all true in $\mathbf{H}$. [*For example, the Axiom of Pairs is true in* $\mathbf{H}$ *provided that, if a, b are hereditarily finite sets, there is a hereditarily finite set c whose only hereditarily finite elements are a and b. This will be true if indeed $\{a, b\}$ is hereditarily finite.*]

(iv) Is $\omega \in \mathbf{H}$?

(v) (For those with B1.1 Logic or equivalent) Show that the Axiom of Infinity is not a consequence (in first order predicate logic) of the Empty Set Axiom, Extensionality, Pairs, Union and Comprehension.

(vi) A set is called *hereditarily countable* if $T(X)$ is a countable set (i.e. is finite or is in bijection with ω). Let $\mathbf{K}$ be the class of hereditarily countable sets. In fact $\mathbf{K}$ is a set. Now $\omega \in \mathbf{K}$. Which of the axioms Extensionality, Empty Set, Pairs, Unions, Comprehension Scheme, Infinity, Powerset hold in $\mathbf{K}$? [*You may use that a countable union of countable sets is countable, though this does not follow from the axioms so far given.*]

(vii) (For those with B1.1 Logic or equivalent) Is it possible to prove the Powerset Axiom from the above axioms (now including Axiom of Infinity)?