

$$y = 1 + \delta_2 \gamma \sim 1 + \delta_2 B \left[ \frac{\varepsilon^{\alpha-1}}{\hat{x}} + \ln(\varepsilon^{1-\alpha} \hat{x}) + \dots \right] \quad \text{for "inner"}$$

$$y \sim 1 - \frac{\varepsilon^\alpha}{\hat{x}} \quad \text{for outer} \quad \therefore \text{Let } \delta_2 = \varepsilon, B = -1$$

$$\therefore 1 + \delta_2 \gamma \sim 1 - \frac{\varepsilon^\alpha}{\hat{x}} - \varepsilon \ln(\varepsilon^{1-\alpha} \hat{x}) + \dots$$

$$\sim 1 - \frac{\varepsilon^\alpha}{\hat{x}} + \underbrace{(1-\alpha)\varepsilon \ln \frac{1}{\varepsilon}}_{\substack{\text{next term} \\ \text{scales with } \varepsilon \ln \frac{1}{\varepsilon}}} - \underbrace{\varepsilon \ln \hat{x}}_{\substack{\text{then scale with } \varepsilon}}$$

$$\therefore \text{We should have written } y \sim y_0(x) + \varepsilon \ln \frac{1}{\varepsilon} y_1(x) + \varepsilon y_2(x) + \dots$$

for the outer ...

Now we can match....

$$(x^2 y_1')' = 0 \quad y_1(x) = C(1 - 1/x) \quad \text{using } y_1(1) = 0.$$

$$\therefore y \sim 1 - \frac{\varepsilon^\alpha}{\hat{x}} + \varepsilon \ln \frac{1}{\varepsilon} C \left(1 - \frac{\varepsilon^\alpha}{\hat{x}}\right) + \varepsilon \left[ A \left(1 - \frac{\varepsilon^\alpha}{\hat{x}}\right) - \ln(\varepsilon^{-\alpha} \hat{x}) - \frac{\varepsilon^\alpha}{\hat{x}} \ln(\varepsilon^{-\alpha} \hat{x}) \right] + \dots \quad \text{in intermediate region}$$

for the outer ...

$$\sim 1 - \frac{\varepsilon^\alpha}{\hat{x}} + \left(\varepsilon \ln \frac{1}{\varepsilon}\right) [C - \alpha] + \dots$$

$$\therefore 1 - \alpha = C - \alpha \quad \text{and } C = 1$$

can now match the inner at leading order

$$\therefore y \sim \left(1 - \frac{1}{x}\right) + \varepsilon \ln \frac{1}{\varepsilon} \left(1 - \frac{1}{x}\right) + O(\varepsilon)$$


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5.2.9

Expansion sequence  $1, \varepsilon \ln \frac{1}{\varepsilon}, \varepsilon, \varepsilon^2 \ln \frac{1}{\varepsilon}, \varepsilon^2 \left( \ln \frac{1}{\varepsilon} \right)^2, \varepsilon^2, \dots$

Van Dyke rule works only if  $\left( \ln \frac{1}{\varepsilon} \right)$  treated as  $O(1)$ .

# 6 Multiple Scales

6.1

## Van der Pol oscillator

$$\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = 0$$

$$\text{with } x = 1, \dot{x} = 0 \text{ at } t = 0$$

$$\text{Let } x \sim x_0(t) + \varepsilon x_1(t) + \dots$$

With regular perturbation expansion

$$x_0(t) = \cos t$$

$$\ddot{x}_1 + x_1 = (1 - x_0^2)\dot{x}_0 \quad \text{with } x_1(0) = \dot{x}_1(0) = 0.$$

$$\therefore \ddot{x}_1 + x_1 = (1 - \cos^2 t)(-\sin t) = \frac{1}{4} \sin 3t - \frac{3}{4} \sin t$$

Will generate resonant terms

$$x_1 = \frac{3}{8} (t \cos t - \sin t) - \frac{1}{32} (\sin 3t - 3 \sin t)$$

$$\therefore x \sim \cos t + \varepsilon \left[ \frac{3}{8} t \cos t + \dots \right] + O(\varepsilon^2)$$

Perturbation expansion breaks down when  $t \sim O(1/\varepsilon)$  as  $x_1$  as large as  $x_0$

Long timescales allow corrections to accumulate.

## Two timescales

$\tau = t$  - fast timescale of oscillation

$T = \varepsilon t$  - slow timescale of amplitude drift

• Look for a solution of the form

$$x(t, \varepsilon) = x(\tau, T, \varepsilon)$$

treating  $\tau, T$  as independent.

$$\frac{d}{dt} = \frac{d\tau}{dt} \frac{\partial}{\partial \tau} + \frac{\partial T}{\partial t} \frac{d}{dT} = \frac{\partial}{\partial \tau} + \varepsilon \frac{\partial}{\partial T}$$

Converting ODE to PDE  
but freedom in  $T$   
dependence used to  
our advantage.

$$\therefore \ddot{x} = x_{tt} = (\partial_\tau + \varepsilon \partial_T)(\partial_\tau + \varepsilon \partial_T)x = x_{\tau\tau} + 2\varepsilon x_{\tau T} + \varepsilon^2 x_{TT}$$

$$\therefore 0 = x_{tt} + \varepsilon(x^2 - 1)x_t + x$$

$$= x_{\tau\tau} + 2\varepsilon x_{\tau T} + \varepsilon^2 x_{TT} + \varepsilon(x^2 - 1)(x_\tau + \varepsilon x_T) + x$$

Expand  $x(\tau, T, \varepsilon) = x_0(\tau, T) + \varepsilon x_1(\tau, T) + \dots$

$O(\varepsilon^0)$

$$\left\{ \begin{array}{l} x_{0\tau\tau} + x_0 = 0 \\ x_0(0) = 1, x_{0\tau}(0) = 0 \end{array} \right\}$$

$$\therefore x_0(\tau, T) = R(T) \cos(\tau + \theta(T))$$

ICS  $R(0) = 1, \theta(0) = 0.$

No other constraints  
on  $R(T), \theta(T)$  at this  
point.

$O(\varepsilon^1)$

$$x_{1\tau\tau} + x_1 = -x_{0\tau}(x_0^2 - 1) - 2x_{0\tau}x_1$$

$$= 2R\theta_T \cos(\tau + \theta) + \left(2R_T + \frac{R^3}{4} - R\right) \sin(\tau + \theta)$$

Will generate  
resonance

$$+ \frac{R^3}{4} \sin 3(\tau + \theta)$$

Will not generate  
resonance

with  $x_1(0) = 0, x_{1\tau}(0) = -x_{0T}(0) = -R_T(0)$

6.3/

$$\therefore \text{Let } R(\tau)\theta_T(\tau) = 0 = (2R_T + R^3/4 - R) \quad \left. \vphantom{\text{Let}} \right\} \begin{array}{l} \text{Known as} \\ \text{"secular"} \\ \text{conditions -} \\ \text{required to avoid} \\ \text{resonance} \end{array}$$

$$\therefore \theta_2 = \text{const, with } \theta(0) = 0 \therefore \theta = 0$$

$$\frac{dR}{d\tau} = \frac{1}{2} [R - R^3/4] \quad \text{with } R(0) = 1 \therefore R = \frac{2}{(1 + 3e^{-\tau})^{1/2}}$$

$$\therefore x(t, \epsilon) = x(\tau, T, \epsilon) = \frac{2}{(1 + 3e^{-\epsilon t})^{1/2}} \cos t + O(\epsilon)$$

Amplitude  $\rightarrow 2$  as  $t \rightarrow \infty$ ,  $\epsilon$  fixed.

### Higher order

We find  $x_1 = -R^3/32 \sin \tau + S(\tau) \sin(\tau + \phi(\tau))$

To find  $S(\tau), \phi(\tau)$  resonant terms are suppressed for  $x_2$  via secular conditions.

However to suppress resonance we must expand with a slow-slow timescale  $T_2 = \epsilon^2 t$ .

### Example

$$\ddot{x} + 2\epsilon \dot{x} + x = 0$$

$$x = A e^{-\epsilon t} \cos(\sqrt{1 - \epsilon^2} t + B)$$

amplitude drift  $t \sim O(1/\epsilon)$

phase drift on  $t \sim O(1/\epsilon^2)$

$$\therefore \text{Let } \tau = t, T_1 = \epsilon t, T_2 = \epsilon^2 t$$

$$d/dt = \partial/\partial \tau + \epsilon \partial/\partial T_1 + \epsilon^2 \partial/\partial T_2$$

} and expand as above

Similarly for higher orders of the Van der Pol oscillator

NB often presented via a complex representation

eg. Van der Pol

$$x_0 = R(\tau) \cos(\tau + \theta(\tau)) = \frac{1}{2} (A e^{i\tau} + \bar{A} e^{-i\tau})$$

$$A = R e^{i\theta}$$

At  $O(\epsilon^1)$

$$x_{1,\tau\tau} + x_1 = -2x_{0,\tau\tau} - (x_0^2 - 1)x_{0,\tau}$$

$$= -i(A_T e^{i\tau} - \bar{A}_T e^{-i\tau}) - \left[ \frac{1}{4} (A e^{i\tau} + \bar{A} e^{-i\tau})^2 - 1 \right] \cdot \frac{i}{2} [A e^{i\tau} - \bar{A} e^{-i\tau}]$$

$$= \left[ -i \left( A_T - \frac{A(4 - |A|^2)}{8} \right) e^{i\tau} + (\text{Complex Conjugate}) \right] + \left[ \text{Non secular terms} \right]$$

$\therefore$  Suppressing resonant terms,  $e^{\pm i\tau}$

$$A_T = \frac{A}{8} (4 - |A|^2) \quad \text{with } A = R e^{i\theta}$$

$$\therefore R_T e^{i\theta} + i R \theta_T e^{i\theta} = \frac{R e^{i\theta}}{8} (4 - R^2)$$

$$\therefore R_T + i R \theta_T = R/8 (4 - R^2) \quad \left. \begin{array}{l} \text{real \& imag} \\ \text{parts} \end{array} \right\} \begin{array}{l} R \theta_T = 0 \\ R_T = R/8 (4 - R^2) \end{array}$$

as before

Note sometimes the slow variable,  $\tau$ , is given the same label as the physical variable  $t$ , so that

$$x_0 = R(\tau) \cos(t + \theta(\tau)) = \frac{1}{2} (A e^{it} + \bar{A} e^{-it}) \text{ above etc.}$$

Homogenization

Example  $\frac{d}{dx} \left( D(x, x/\varepsilon) \frac{du}{dx} \right) = f(x) \quad 0 < x < 1 \quad (*)$   
 $u(0) = a, \quad u(1) = b \quad a, b \in \mathbb{R}^+$

$D, f$  are smooth, with  $0 < D_-(x) < D(x, X) < D_+(x)$ , with  $D_{\pm}$  continuous.

Question Can (\*) be approximated by  $\frac{d}{dx} (\bar{D}(x) \frac{du}{dx}) = f(x)$   
 $u(0) = a, \quad u(1) = b$

for an averaged function  $\bar{D}(x) \leftarrow$  does not contain fast  $\varepsilon$  variation.

Multiple Scales

Let  $u(x, \varepsilon) = u(x, X, \varepsilon)$  with  $X = x/\varepsilon$   
not relabelling as separate variable fast variable

$$\frac{d}{dx} = \frac{\partial}{\partial x} + \frac{1}{\varepsilon} \frac{\partial}{\partial X}$$

$$\therefore (\partial_x + \frac{1}{\varepsilon} \partial_X) \left[ D(x, X) (\partial_x + \frac{1}{\varepsilon} \partial_X) u \right] = f(x)$$

$$\therefore (\varepsilon \partial_x + \partial_X) \left[ D(x, X) (\varepsilon \partial_x + \partial_X) u \right] = \varepsilon^2 f(x)$$

Let  $u \sim u_0(x, X) + \varepsilon u_1(x, X) + \dots$

Also assume  
 $u_0, u_1, u_2, \dots$  bounded  
 for ALL  $X$

$O(\varepsilon^0)$   $(D(x, X) u_{0x})_X = 0$

$O(\varepsilon^1)$   $(D(x, X) [u_{1x} + u_{0x}])_X + (D(x, X) u_{0x})_x = 0$

$O(\varepsilon^2)$   $(D(x, X) [u_{2x} + u_{1x}])_X + (D(x, X) [u_{1x} + u_{0x}])_x = f(x)$

$O(\varepsilon^0)$ 

$$D u_{0x} = c_1(x)$$

$$\therefore u_0 = c_2(x) + c_1(x) \underbrace{\int_0^x \frac{ds}{D(x,s)}}_{\text{ord}(x) \text{ as } X \rightarrow \infty} \quad c_1, c_2 \text{ arbitrary}$$

$$\text{as } \frac{1}{D_+(x)} \leq \frac{1}{D} \leq \frac{1}{D_-(x)}$$

 $u_0$  bounded

$$\therefore u_0 = c_2(x) \quad \text{and } c_1(x) = 0 \text{ all } x.$$

$$\therefore \text{We write } u_0 = u_0(x).$$

 $O(\varepsilon^1)$ 

$$\left( D(x,x) [u_{0x} + u_{1x}] \right)_x = 0$$

$$D[u_{0x} + u_{1x}] = d_1(x) \quad (++)$$

$$\therefore u_1 = d_1(x) \int_0^x \frac{ds}{D(x,s)} - X u_{0x} + d_2(x)$$

↑ Blow up ↑  
as  $X \rightarrow \infty$ , with both  $\text{ord}(x)$ .  
Hence  $D_H(x)$  exists.

$$\therefore \text{let } d_1(x) = \left[ \lim_{X \rightarrow \infty} \frac{X}{\int_0^x \frac{ds}{D(x,s)}} \right] u_{0x} \equiv D_H(x) u_{0x}$$

 $O(\varepsilon^2)$ 

$$\left( D(x,x) [u_{2x} + u_{1x}] \right)_x = f(x) - d_{1x} \quad \text{using } (++)$$

$$\therefore D(x,x) [u_{2x} + u_{1x}] = e_1(x) + (f(x) - d_{1x}) X$$

$$\therefore u_{2x} = \frac{e_1(x)}{D(x,x)} + (f(x) - d_{1x}) \frac{X}{D(x,x)} - u_{1x}$$

$$\therefore u_2 = e_1(x) \underbrace{\int_0^x \frac{ds}{D(x,s)}}_{\text{ord}(x) \text{ as } X \rightarrow \infty} + (f(x) - d_{1x}) \underbrace{\int_0^x \frac{s}{D(x,s)} ds}_{\text{ord}(x^2) \text{ as } X \rightarrow \infty} - \underbrace{\int_0^x u_{1x} ds}_{\text{ord}(x) \text{ as } X \rightarrow \infty}$$



NB

$$u_1 = \underbrace{\left\{ \left[ \lim_{P \rightarrow \infty} \left\{ \frac{P}{\int_0^P \frac{ds}{D(x,s)}} \right\} \int_0^X \frac{ds}{D(s,x)} \right] - X \right\}}_{\text{ord}(1) \text{ as } X \rightarrow \infty} u_{0,x} + d_2(x)$$

$$\int_0^X u_{1,x} ds = \int_0^X \text{ord}(1) u_{0,x} + d_2(x) ds = \text{ord}(X) \text{ as } X \rightarrow \infty$$

$\therefore$  For  $u_2(x, X)$  to be bounded, we must have

$$d_{1,x} = f(x)$$

$$\therefore \frac{d}{dx} \left( D_H(x) \frac{du_0}{dx} \right) = f(x) \quad (*)$$

$$\text{with } D_H(x) = \lim_{X \rightarrow \infty} \left[ \frac{X}{\int_0^X \frac{ds}{D(x,s)}} \right]$$

$$u_0(0) = a, \quad u_0(1) = b$$

(\*) is a "homogenized" ODE

NB If  $D(x,s)$  is periodic, say with period 1,  $D_H$  simplifies by taking  $X \in \mathbb{N}$

$$D_H = \lim_{X \rightarrow \infty} \left[ \frac{X}{X \int_0^1 \frac{ds}{D(x,s)}} \right] = \frac{1}{\int_0^1 \frac{ds}{D(x,s)}}$$

In higher dimensional problems, periodicity often has to be assumed to make progress