C2.7: CATEGORY THEORY

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Introduction

Historically, people first studied numbers and then realized that their collection forms a set: e.g. the set of natural numbers, rational numbers etc. The next natural question is as follows: what do sets or groups form? For two numbers they can either be equal or not equal. On the other hand for sets, besides equality one also has a notion of isomorphism of sets: the sets $\{1, 2\}$ and $\{2, 3\}$ are not equal but isomorphic. Thus, sets form a more complicated structure, which we will call a category.

Categorical methods allow one to abstract from the precise context and prove statements valid for any kinds of objects: sets, groups, vector spaces etc. Often in concrete contexts the precise construction might be complicated but it may satisfy some universal property which is easy to use in practice and which determines the construction uniquely. Let us give some examples of this phenomenon.

Given two vector spaces V, W over a field k one can form their tensor product $V \otimes W$. The precise definition is rather involved: $V \otimes W$ is the vector space spanned by elements of the form $v \otimes w$ for $v \in V$ and $w \in W$ modulo the relations

- $(\lambda v) \otimes w = v \otimes (\lambda w)$ for $\lambda \in k$,
- $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$,
- $v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$.

However, it satisfies a *universal property*: a linear map $f\colon V \otimes W \to Z$ to another vector space Z is the same as a map of sets $V \times W \to Z$ which is bilinear in each variable:

$$f(\lambda_1 v_1 + \lambda_2 v_2, w) = \lambda_1 f(v_1, w) + \lambda_2 f(v_2, w)$$

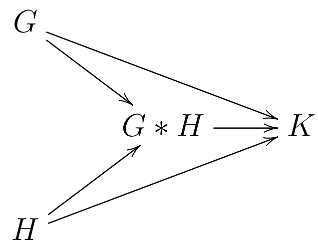
and similarly for the second slot.

We will later show in the course that this universal property determines the tensor product uniquely. Let us give an easy example of how one works with universal properties by showing that $V \otimes k \cong V$. Indeed, a map $V \otimes k \to Z$ is the same as a map of sets $V \times k \to Z$ which is bilinear. Linearity in the second variable shows that the map is uniquely specified by its value on $V \times \{1\}$. Therefore, such a bilinear map is the same as a linear map $V \to Z$.

Here is another example. Given two groups G, H one can consider their free product $G * H$. Its underlying set consists of formal words $g_1 g_2 ... g_n$ for g_i either in G or H modulo the relations that one can multiply two adjacent elements if they are both in G or in H. The product is given by concatenation. Note that we have maps $G \to G * H$ and $H \to G * H$.

The universal property of the free product is the following. Given any group K together with homomorphisms $G \to K$ and $H \to K$ one has a unique homomorphism $G * H \to K$

such that the diagram



commutes. We will later see that this construction can be generalized to objects of other categories than just groups.

Literature. These are lecture notes from an introductory course on category theory taught in Michaelmas term 2015. Due to time constraints we have not covered many topics that could be found in a different introductory course. Besides the comprehensive classic Mac Lane's book [McL], let us also recommend the reader to check out [Awo], [Lei] and [Rie].

The following topics could be found in some other category theory courses:

- Monoidal, braided monoidal and symmetric monoidal categories,
- Operads,
- Kan extensions,
- 2-categories, higher categories.

The reader is invited to consult other sources for these (in particular, Mac Lane's book).

1. Basic definitions

1.1. **Categories.** The main definition in this course is that of a category. We will use the word "collection" to refer to certain large sets. These set-theoretic issues will be discussed in the next section, but for now one can replace the word "collection" with "set".

Definition 1.1. A *category* $\mathcal{C}$ consists of the following data:

(1) a collection $\mathrm{ob}\,\mathcal{C}$ of *objects* of $\mathcal{C}$,
(2) for every two objects $x, y \in \mathrm{ob}\,\mathcal{C}$ a collection $\mathrm{Hom}_{\mathcal{C}}(x, y)$ of *morphisms*,
(3) the identity morphism $\mathrm{id}_x \in \mathrm{Hom}_{\mathcal{C}}(x, x)$ for every object $x \in \mathrm{ob}\,\mathcal{C}$,
(4) the composition map
$$\circ\colon \mathrm{Hom}_{\mathcal{C}}(y, z) \times \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{C}}(x, z)$$
for every triple of objects $x, y, z \in \mathrm{ob}\,\mathcal{C}$.

These have to satisfy the following two axioms:

(1) (Units). For any two objects $x, y \in \mathrm{ob}\,\mathcal{C}$ and any morphism $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$ one has
$$f \circ \mathrm{id}_x = f$$
and
$$\mathrm{id}_y \circ f = f.$$
(2) (Associativity). For any four objects $x, y, z, v \in \mathrm{ob}\,\mathcal{C}$ and any three morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$, $g \in \mathrm{Hom}_{\mathcal{C}}(y, z)$, $h \in \mathrm{Hom}_{\mathcal{C}}(z, v)$ one has
$$(h \circ g) \circ f = h \circ (g \circ f).$$

We will use a shorthand $x \in \mathcal{C}$ for $x \in \operatorname{ob}\mathcal{C}$ and often omit the subscript in Hom when the category in question is clear. Note that some people use $\mathcal{C}(x, y)$ for what we denote in the course by $\operatorname{Hom}_{\mathcal{C}}(x, y)$. We will also write $\operatorname{End}_{\mathcal{C}}(x)$ for $\operatorname{Hom}_{\mathcal{C}}(x, x)$ and refer to morphisms from an object to itself as *endomorphisms*.

Informally one thinks of a category as a collection of objects connected by composable arrows: an arrow between two objects specifies a morphism.

Example. Consider a category $\mathcal{C}$ with a single object, traditionally written $*$. Then $\operatorname{End}_{\mathcal{C}}(*)$ forms what's known as a *monoid* (and thus, a category can be thought of as a monoid with many objects). Let $\operatorname{End}_{\mathcal{C}}(*) = A$, then we denote such a category by $*/A$.

If we furthermore assume existence of inverses we get the definition of a group.

Given a category $\mathcal{C}$ we have its *opposite* category $\mathcal{C}^{op}$. They have the same objects and $\operatorname{Hom}_{\mathcal{C}^{op}}(x, y) = \operatorname{Hom}_{\mathcal{C}}(y, x)$.

1.2. **Set-theoretic issues.** This section can be skipped and will not be discussed in the lectures. A more detailed account on Grothendieck universes can be found in [Low].

We would like to say that sets form a category Set with objects of Set being sets and morphisms being maps of sets. Informally, all axioms seem to be satisfied. The problem is that the set of all sets does not exist; there are several ways to get around this depending on one's axioms of set theory. For some choices of such axioms the collection of all sets forms a class, but this has no meaning in the Zermelo–Fraenkel axiomatics.

We will adopt the Tarski–Grothendieck axioms of set theory.

Definition 1.2. A *Grothendieck universe* U is a set satisfying the following properties:

(1) If $x \in y$ and $y \in U$, then $x \in U$.
(2) If $x \in U$ and $y \in U$, then $\{x, y\} \in U$.
(3) If $x \in U$, then the power set of x is in U.
(4) If $x \in U$ and $f\colon x \to U$ is a map, then $\cup_{i \in x} f(i) \in U$.

We will often refer to Grothendieck universes simply as universes. One way to formulate the Tarski–Grothendieck axioms is to start with the axioms of Zermelo–Fraenkel with the axiom of choice and add the following axiom known as the *Tarski axiom.*

Axiom. For every set X there is a universe U containing X.

We call a set X *U-small* for some universe U if $X \in U$. In this axiomatics Russel's paradox that the set of all sets doesn't exist is avoided since a universe U is not a member of itself. In other words, the universe U itself is not U-small.

Fix a universe U. Then we can define the category of U-small sets Set_U. Its objects form a set, but this set is not U-small. However, it follows from the axioms that the set of objects of Set_U is V-small for some bigger universe V.

Here are some important definitions to keep in mind.

Definition 1.3. A category $\mathcal{C}$ is *U-small* if $\operatorname{ob}\mathcal{C}$ is a U-small set and $\operatorname{Hom}_{\mathcal{C}}(x, y)$ is a U-small set for any $x, y \in \mathcal{C}$.

Definition 1.4. A category $\mathcal{C}$ is *U-locally small* if $\operatorname{ob}\mathcal{C} \subseteq U$ and for any two objects $x, y \in \mathcal{C}$ the set $\operatorname{Hom}_{\mathcal{C}}(x, y)$ is U-small.

As we mentioned before, any category is V-small for a large enough universe V. In the course we will fix a universe implicitly and talk about small and large categories with respect to that universe.

1.3. **Functors.** Whenever one encounters a structure, one should ask what are the relations between objects possessing that kind of structure so that one can define the category of such objects, with morphisms given by maps preserving the structure. For categories themselves this gives us the notion of a functor.

Definition 1.5. A *functor* $F\colon \mathcal{C} \to \mathcal{D}$ between two categories $\mathcal{C}$ and $\mathcal{D}$ consists of the following data:

- a map $\operatorname{ob}\mathcal{C} \to \operatorname{ob}\mathcal{D}$ of sets (also denoted by F),
- for any two objects $x, y \in \mathcal{C}$ a map of sets $\operatorname{Hom}_{\mathcal{C}}(x, y) \to \operatorname{Hom}_{\mathcal{D}}(F(x), F(y))$ (again denoted by F).

These have to satisfy the following axioms:

(1) (Unit). For any object $x \in \mathcal{C}$ we have $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$.
(2) (Composition). For any three objects $x, y, z \in \mathcal{C}$ and morphisms $f \in \operatorname{Hom}_{\mathcal{C}}(x, y)$, $g \in \operatorname{Hom}_{\mathcal{C}}(y, z)$ one has
$$F(g \circ f) = F(g) \circ F(f).$$

If A and B are two monoids, then a functor $*/A \to */B$ is the same as a homomorphism of monoids. (If readers are scared by the word "monoid", they can replace it by "group" with the same result.)

We will say a functor $F\colon \mathcal{C} \to \mathcal{D}$ is *faithful* if the map $\operatorname{Hom}_{\mathcal{C}}(x, y) \to \operatorname{Hom}_{\mathcal{D}}(F(x), F(y))$ is injective for any objects x and y. We say F is *full* if this map is surjective for any x, y, and we say F is *fully faithful* if it is both full and faithful.

We say that F is a *contravariant* functor from $\mathcal{C}$ to $\mathcal{D}$ if it is a functor $\mathcal{C}^{op} \to \mathcal{D}$. A functor $\mathcal{C} \to \mathcal{D}$ is also referred to as a covariant functor.

1.4. **Examples.** We have already mentioned one example of a category, which is the category of (small) sets. Let us give some more examples.

(1) Groups form a category Grp with morphisms given by homomorphisms of groups. The notation Hom originates from this example.
(2) One can restrict groups to be abelian, these form a category Ab.
(3) If k is a field, k-vector spaces form a category Vect_k with morphisms given by linear maps.
(4) (Small) categories themselves form a category Cat with morphisms given by functors.
(5) A set X can be regarded as a category $\mathcal{C}$ with $\operatorname{ob}\mathcal{C} = X$ and $\operatorname{Hom}_{\mathcal{C}}(x, y) = \varnothing$ for $x \neq y$ and $\operatorname{End}_{\mathcal{C}}(x) = \{\mathrm{id}_x\}$. We call such a category a *discrete category.*

Definition 1.6. A morphism $f \in \operatorname{Hom}(x, y)$ in a category is an *isomorphism* if there is a morphism $f^{-1} \in \operatorname{Hom}(y, x)$ such that $f^{-1} \circ f = \mathrm{id}_x$ and $f \circ f^{-1} = \mathrm{id}_y$. We also say that that f is *invertible.*

The standard argument from group theory shows that the morphism f^{-1} is unique if it exists; thus being an isomorphism is a property of a morphism rather than extra data.

If there is an isomorphism between two objects $x, y \in \mathcal{C}$ we will write $x \cong y$.

Definition 1.7. A category $\mathcal{C}$ is called a *groupoid* if every morphism is invertible.

The reason such categories are called groupoids is given by the following observation. Consider a groupoid with a single object. The data needed to specify such a groupoid is the monoid of endomorphisms in which every object has an inverse. In other words, the monoid of endomorphisms is actually a group in this case. So, groupoids can be thought of as "multi-object" versions of groups.

We will say that a groupoid is *connected* if any two objects are isomorphic.

Let us now give some examples of functors.

1. Given an abelian group one can forget that it is abelian which gives a functor $\mathrm{Ab} \to \mathrm{Grp}$. It is fully faithful.
2. The abelianization $G \mapsto G/[G, G]$ gives a functor $\mathrm{Grp} \to \mathrm{Ab}$. It is neither full nor faithful.
3. Given a set we can regard it as a discrete category. This gives a fully faithful functor $\mathrm{Set} \to \mathrm{Cat}$.

1.5. **Natural transformations.** Consider the category of sets Set. Given two sets $x, y \in \mathrm{Set}$ we can ask two different questions:

1. Is $x = y$?
2. Is $x \cong y$?

Let us remind the reader that the axiom of extensionality implies that the sets x and y are equal iff they have the same objects.

Clearly, if $x = y$ then $x \cong y$ as we may take the identity morphism to represent the isomorphism. However, there are isomorphic sets which are not equal: for instance, the group of units in $\mathbb{Z}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ but they are not equal. In general the philosophy one usually takes in category theory is that questions of the first kind are unnatural[1]in category theory and should always be replaced by questions of the second kind.

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and consider two functors $F, G\colon \mathcal{C} \to \mathcal{D}$. The statement $F = G$ means the following:

1. For every object $x \in \mathcal{C}$ we have $F(x) = G(x)$.
2. For every morphism $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$ we have $F(f) = G(f)$ in $\mathrm{Hom}_{\mathcal{D}}(F(x), F(y))$.

In particular, the statement $F = G$ implies an equality of certain objects, which is an unnatural question. We will formulate a "natural" version of equality for functors which is called a natural isomorphism.

Definition 1.8. Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be two functors. A *natural transformation* $\eta\colon F \Rightarrow G$ consists of morphisms $\eta_x \in \mathrm{Hom}_{\mathcal{D}}(F(x), G(x))$ for every object $x \in \mathcal{C}$ such that the diagram

$$\begin{array}{ccc} F(x) & \xrightarrow{F(f)} & F(y) \\ \downarrow{\scriptstyle \eta_x} & & \downarrow{\scriptstyle \eta_y} \\ G(x) & \xrightarrow{G(f)} & G(y) \end{array}$$

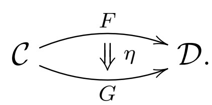
commutes for every morphism $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$.

[1]On the category theory wiki http://ncatlab.org these kind of questions are referred to as "evil".

We say a natural transformation $\eta\colon F \Rightarrow G$ is a *natural isomorphism* if the morphisms η_x are isomorphisms for any $x \in \mathcal{C}$.

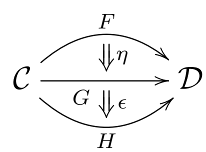
Given two categories $\mathcal{C}, \mathcal{D}$ one can construct a category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ of functors between $\mathcal{C}$ and $\mathcal{D}$: its objects are functors $\mathcal{C} \to \mathcal{D}$ and morphisms are given by natural transformations. Then natural isomorphisms are simply isomorphisms in the functor category.

1.6. **Composing natural transformations.** Given two categories $\mathcal{C}, \mathcal{D}$ and a pair of functors $F, G\colon \mathcal{C} \to \mathcal{D}$ we depict a natural transformation $\eta\colon F \Rightarrow G$ in the following way:



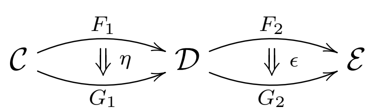
There are two main ways one can compose natural transformations: horizontal and vertical.

(1) Consider a diagram



We have a *vertical composition* $F \Rightarrow H$ which has components given by the composites $F(x) \xrightarrow{\eta_x} G(x) \xrightarrow{\epsilon_x} H(x)$ for all $x \in \mathcal{C}$.

(2) Consider a diagram

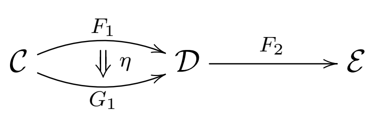


We have a *horizontal composition* $F_2F_1 \Rightarrow G_2G_1$ which has components

$$F_2F_1(x) \xrightarrow{F_2(\eta_x)} F_2G_1(x) \xrightarrow{\epsilon_{G_1(x)}} G_2G_1(x)$$

for every $x \in \mathcal{C}$. Note that we could have used the composition of ϵ and η in a different order; the two expressions coincide due to naturality of ϵ and η.

(3) Finally, one can also compose natural transformations with functors. Consider a diagram



Then we have a *whiskering* of η and F_2 being the natural transformation $F_2F_1 \Rightarrow F_2G_1$ whose components are $F_2F_1(x) \xrightarrow{F_2(\eta_x)} F_2G_1(x)$ for every $x \in \mathcal{C}$. Clearly, whiskering is an example of a horizontal composition where $G_2 = F_2$ and $\epsilon = \mathrm{id}$. Similarly, we can precompose a natural transformation with a functor which is a horizontal composition with $\eta = \mathrm{id}$.

1.7. **Equivalences of categories.** In the category Cat of categories an isomorphism $F\colon \mathcal{C} \to \mathcal{D}$ is an isomorphism of sets $\mathrm{ob}\,\mathcal{C} \to \mathrm{ob}\,\mathcal{D}$ and an isomorphism $\mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(F(x), F(y))$ for each $x, y \in \mathrm{ob}\,\mathcal{C}$. We are going to define a weaker notion of equivalence of categories.

Definition 1.9. An *equivalence* of categories $\mathcal{C}, \mathcal{D}$ is a pair of functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ together with natural isomorphisms $e\colon \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ and $\epsilon\colon FG \Rightarrow \mathrm{id}_{\mathcal{D}}$.

Definition 1.10. An *adjoint equivalence* of categories $\mathcal{C}, \mathcal{D}$ is an equivalence (F, G, e, ϵ) satisfying the following axioms:

(1) The composite natural transformation

$$F \cong F \circ \mathrm{id}_{\mathcal{C}} \overset{\mathrm{id}_F \circ e}{\Rightarrow} FGF \overset{\epsilon \circ \mathrm{id}_F}{\Rightarrow} \mathrm{id}_{\mathcal{D}} \circ F \cong F$$

is the identity natural transformation on F.

(2) The composite natural transformation

$$G \cong \mathrm{id}_{\mathcal{C}} \circ G \overset{e \circ \mathrm{id}_G}{\Rightarrow} GFG \overset{\mathrm{id}_G \circ \epsilon}{\Rightarrow} G \circ \mathrm{id}_{\mathcal{D}} \cong G$$

is the identity natural transformation on G.

Now we get three notions of sameness for categories:

(1) Isomorphism of categories,
(2) Equivalence of categories,
(3) Adjoint equivalence of categories.

Clearly, an isomorphism of categories is an adjoint equivalence, but the two notions of equivalences are much more flexible compared to the strict notion of isomorphism which does not appear in practice. Moreover, we will show that the two notions of equivalences are the same.

Let us fix a functor $F\colon \mathcal{C} \to \mathcal{D}$ and consider the following category Equiv_F. Its objects are functors $G\colon \mathcal{D} \to \mathcal{C}$ together with natural isomorphisms e and ϵ such that (F, G, e, ϵ) is an adjoint equivalence. A morphism $(G_1, e_1, \epsilon_1) \to (G_2, e_2, \epsilon_2)$ consists of a natural transformation $f\colon G_1 \Rightarrow G_2$ making the diagrams

$$\begin{array}{ccc} & \mathrm{id}_{\mathcal{C}} & \\ {}^{e_1}\swarrow & & \searrow^{e_2} \\ G_1F & \overset{f \circ \mathrm{id}_F}{\Longrightarrow} & G_2F \end{array}$$

and

$$\begin{array}{ccc} & \mathrm{id}_{\mathcal{D}} & \\ {}^{\epsilon_1}\nearrow & & \nwarrow^{\epsilon_2} \\ FG_1 & \overset{\mathrm{id}_F \circ f}{\Longrightarrow} & FG_2 \end{array}$$

commute.

Proposition 1.11. *Any two objects* (G_1, e_1, ϵ_1) *and* (G_2, e_2, ϵ_2) *of* Equiv_F *are isomorphic and, moreover, this isomorphism is unique.*

Proof. To prove the first statement, consider the composite natural isomorphism $f : G_1 \Rightarrow G_2$ given by

$$G_1 \cong \mathrm{id}_{\mathcal{C}} \circ G_1 \overset{e_2 \circ \mathrm{id}_{G_1}}{\Rightarrow} G_2 F G_1 \overset{\mathrm{id}_{G_2} \circ \epsilon_1}{\Rightarrow} G_2.$$

To show that the first diagram commutes, consider the composite natural transformation

$$\mathrm{id} \overset{e_1}{\Rightarrow} G_1 F \overset{e_2 \circ \mathrm{id}_{G_1} \circ \mathrm{id}_F}{\Rightarrow} G_2 F G_1 F \overset{\mathrm{id}_{G_2} \circ \epsilon_1 \circ \mathrm{id}_F}{\Rightarrow} G_2 F.$$

Since e_1 and e_2 are natural transformations, we can alternatively write this as

$$\mathrm{id} \overset{e_2}{\Rightarrow} G_2 F \overset{\mathrm{id}_{G_2} \circ \mathrm{id}_F \circ e_1}{\Rightarrow} G_2 F G_1 F \Rightarrow G_2 F.$$

But then using the compatibility of e_1 and ϵ_1 we get that the composite natural transformation is e_2 as required. The commutativity of the other diagram is checked in the same way. Therefore, any two objects are isomorphic.

To prove the second statement, consider a morphism $f \in \mathrm{Hom}((G_1, e_1, \epsilon_1), (G_2, e_2, \epsilon_2))$ and take any $x \in \mathcal{C}$ and $y \in \mathcal{D}$. Then the commutativity of the diagrams implies that

$$f_{F(x)} \circ e_{1,x} = e_{2,x}$$

and

$$\epsilon_{2,y} \circ F(f_y) = \epsilon_{1,y}.$$

Equivalently,

$$f_{F(x)} = e_{2,x} \circ e_{1,x}^{-1}, \quad F(f_y) = \epsilon_{2,y}^{-1} \circ \epsilon_{1,y}.$$

Applying G_2 to the second equality we get

$$G_2 F(f_y) = G_2(\epsilon_{2,y})^{-1} \circ G_2(\epsilon_{1,y}).$$

Since e_2 is a natural isomorphism, we get

$$e_{2,G_2(y)} \circ f_y = G_2(\epsilon_{2,y})^{-1} \circ G_2(\epsilon_{1,y}) \circ e_{2,G_1(y)}$$

which implies that

$$f_y = G_2(\epsilon_{1,y}) \circ e_{2,G_1(y)}.$$

In other words, f is determined uniquely. □

This proposition implies that given a functor $F : \mathcal{C} \to \mathcal{D}$ the other functor in the quadruple (F, G, e, ϵ) is unique up to a unique natural isomorphism if it exists. Therefore, being an adjoint equivalence is a property of a functor $F : \mathcal{C} \to \mathcal{D}$ rather than extra data.

Let us now give a more direct way of showing that a given functor underlines an equivalence of categories.

Definition 1.12. A functor $F : \mathcal{C} \to \mathcal{D}$ is *essentially surjective* if for every object $d \in \mathcal{D}$ there is $c \in \mathcal{C}$ such that $F(c) \cong d$.

Note that we do not assume that such a c is unique or that the assignment $d \mapsto c$ is functorial in any way.

Theorem 1.13. *The following three properties of a functor $F : \mathcal{C} \to \mathcal{D}$ are equivalent:*

(1) *F is a part of an equivalence of categories,*
(2) *F is fully faithful and essentially surjective,*
(3) *F is a part of an adjoint equivalence of categories.*

Proof.

- **(3)** $\Rightarrow$ **(1)** is obvious.
- **(1)** $\Rightarrow$ **(2)**.

  Let (F, G, e, ϵ) be an equivalence. For any $d \in \mathcal{D}$ we have $F(G(d)) \cong d$, so F is essentially surjective.

  For any objects $x, y \in \mathcal{C}$ the composite

  $$\mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(F(x), F(y)) \to \mathrm{Hom}_{\mathcal{C}}(GF(x), GF(y))$$

  is an isomorphism, so F is faithful.

  The same reasoning shows that G is faithful, so F has to be full.
- **(2)** $\Rightarrow$ **(3)**.

  Conversely, suppose F is fully faithful and essentially surjective. Consider the following category $\mathcal{CD}$:
  - Its objects are objects $c \in \mathcal{C}$ and $d \in \mathcal{D}$ together with an isomorphism $i\colon F(c) \xrightarrow{\sim} d$.
  - A morphism from (c_1, d_1, i_1) to (c_2, d_2, i_2) is a morphism $f_c \in \mathrm{Hom}_{\mathcal{C}}(c_1, c_2)$ and a morphism $f_d \in \mathrm{Hom}_{\mathcal{D}}(d_1, d_2)$ making the diagram

    $$\begin{array}{ccc} F(c_1) & \xrightarrow{i_1} & d_1 \\ {\scriptstyle F(f_c)}\downarrow & & \downarrow{\scriptstyle f_d} \\ F(c_2) & \xrightarrow{i_2} & d_2 \end{array}$$

    commute.

  We have the following natural functors:
  - The forgetful functor $P_{\mathcal{C}}\colon \mathcal{CD} \to \mathcal{C}$ given by $P_{\mathcal{C}}(c, d, i) = c$.
  - The forgetful functor $P_{\mathcal{D}}\colon \mathcal{CD} \to \mathcal{D}$ given by $P_{\mathcal{D}}(c, d, i) = d$.
  - The inclusion functor $I_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{CD}$ given by $I_{\mathcal{C}}(c) = (c, F(c), \mathrm{id}_{F(c)})$.

  The composite $P_{\mathcal{C}} I_{\mathcal{C}}$ is the identity functor on $\mathcal{C}$. We also have a natural isomorphism $I_{\mathcal{C}} P_{\mathcal{C}} \Rightarrow \mathrm{id}_{\mathcal{CD}}$. For an object $(c, d, i) \in \mathcal{CD}$ the morphism

  $$I_{\mathcal{C}} P_{\mathcal{C}}(c, d, i) = (c, F(c), \mathrm{id}_{F(c)}) \to (c, d, i)$$

  is given by $\mathrm{id}_c\colon c \to c$ and $i\colon F(c) \to d$. It is not difficult to see that $P_{\mathcal{C}}$ and $I_{\mathcal{C}}$ therefore constitute an equivalence.

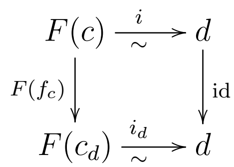
  We are now going to define a functor $I_{\mathcal{D}}\colon \mathcal{D} \to \mathcal{CD}$ such that the pair $(P_{\mathcal{D}}, I_{\mathcal{D}})$ is an equivalence.

  For every object $d \in \mathcal{D}$ consider the set S_d consisting of objects $c \in \mathcal{C}$ together with an isomorphism $i\colon F(c) \xrightarrow{\sim} d$. Since F is essentially surjective, every set S_d is nonempty. Using the axiom of choice we can therefore choose elements $(c_d, d, i_d) \in S_d$ for every $d \in \mathcal{D}$. We define the functor $I_{\mathcal{D}}$ by sending $d \in \mathcal{D}$ to $(c_d, d, i_d) \in \mathcal{CD}$. A morphism $f\colon d_1 \to d_2$ determines uniquely a morphism $\tilde{f}_c\colon F(c_1) \to F(c_2)$ making the diagram

  $$\begin{array}{ccc} F(c_{d_1}) & \underset{\sim}{\xrightarrow{i_{d_1}}} & d_1 \\ {\scriptstyle \tilde{f}_c}\downarrow & & \downarrow{\scriptstyle f} \\ F(c_{d_2}) & \underset{\sim}{\xrightarrow{i_{d_2}}} & d_2 \end{array}$$

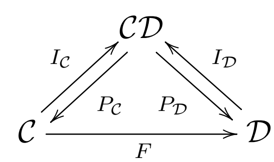
commute. By fully faithfulness of F, there is a unique morphism f_c such that $F(f_c) = \tilde{f}_c$. This concludes the definition of the functor $I_{\mathcal{D}}$.

We have $P_{\mathcal{D}} I_{\mathcal{D}} = \mathrm{id}_{\mathcal{D}}$. The natural isomorphism $I_{\mathcal{D}} P_{\mathcal{D}} \Rightarrow \mathrm{id}_{\mathcal{CD}}$ is defined as follows. $I_{\mathcal{D}} P_{\mathcal{D}}(c, d, i) = (c_d, d, i_d)$. By fully faithfulness of F we can find a unique morphism $f_c \colon c \to c_d$ making the diagram



commute. Again it is easy to show that the pair $(P_{\mathcal{D}}, I_{\mathcal{D}})$ is an adjoint equivalence.

To conclude the proof of the theorem, consider the diagram



The diagram commutes strictly (i.e. $P_{\mathcal{D}} I_{\mathcal{C}} = F$) and every pair of functors in the diagram is an adjoint equivalence which implies that F itself is an adjoint equivalence.

□

Since the notions of adjoint equivalence and simply equivalence are essentially the same by the previous theorem, from now on we will simply refer to a single functor constituting an equivalence. In practice one often just checks the property (2) in Theorem 1.13.

2. Adjoint functors

2.1. **Definition via units.** If one has a map of sets $f \colon X \to Y$, the only natural map that goes in the other direction is the inverse $f^{-1} \colon Y \to X$ if it exists. For categories we can talk about adjoint equivalences which is a notion analogous to the notion of inverses of maps of sets. However, we can also relax the definition of adjoint equivalences to obtain more general class of "inverses" of categories which are known as adjoint functors. Adjoint functors are central constructions in category theory and we will give several equivalent definitions.

Let $\mathcal{C}, \mathcal{D}$ be two categories.

Definition 2.1. A pair of functors $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{D} \to \mathcal{C}$ are *adjoint* if there are natural transformations $e \colon \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ and $\epsilon \colon FG \Rightarrow \mathrm{id}_{\mathcal{D}}$ satisfying the following axioms:

(1) The composite
$$F \overset{\mathrm{id}_F \circ e}{\Rightarrow} FGF \overset{\epsilon \circ \mathrm{id}_F}{\Rightarrow} F$$
is the identity natural transformation,

(2) The composite
$$G \overset{e \circ \mathrm{id}_G}{\Rightarrow} GFG \overset{\mathrm{id}_G \circ \epsilon}{\Rightarrow} G$$
is the identity natural transformation.

If there is an adjoint pair (F, G) we say that F is a *left adjoint* and G is a *right adjoint* and we write $F \dashv G$. The natural transformation e is known as the *unit* of the adjunction and ϵ is the *counit* of the adjunction.

Clearly, an adjoint equivalence is an instance of an adjunction where the natural transformations e, ϵ are isomorphisms. Note, however, that the definition of an adjoint equivalence is symmetric under the exchange of F and G while a general adjunction is not.

We will prove the following theorem in the next section.

Theorem 2.2. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. If a right adjoint $G\colon \mathcal{D} \to \mathcal{C}$ exists, it is unique up to a unique natural isomorphism. The same result holds true for the left adjoint.*

2.2. **Definition via natural isomorphisms.** Suppose $F \dashv G$ and consider a morphism $f\colon F(x) \to y$ for $x \in \mathcal{C}$ and $y \in \mathcal{D}$. The composite

$$x \stackrel{e_x}{\to} GF(x) \stackrel{G(f)}{\to} G(y)$$

gives a morphism $x \to G(y)$ in $\mathcal{C}$. This defines a map of sets $\mathrm{Hom}_{\mathcal{D}}(F(x), y) \to \mathrm{Hom}_{\mathcal{C}}(x, G(y))$. Using the counit one can define a map $\mathrm{Hom}_{\mathcal{C}}(x, G(y)) \to \mathrm{Hom}_{\mathcal{D}}(F(x), y)$ which is the inverse. In other words, we have defined an isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(F(x), y) \cong \mathrm{Hom}_{\mathcal{C}}(x, G(y)).$$

This isomorphism is natural in the two variables x and y in the following sense. Consider a morphism $y \to y'$ in $\mathcal{D}$. Then we get maps $\mathrm{Hom}_{\mathcal{D}}(F(x), y) \to \mathrm{Hom}_{\mathcal{D}}(F(x), y')$ and $\mathrm{Hom}_{\mathcal{C}}(x, G(y)) \to \mathrm{Hom}_{\mathcal{C}}(x, G(y'))$ given by post-composition. The naturality is expressed by the commutativity of the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{D}}(F(x), y) & \longrightarrow & \mathrm{Hom}_{\mathcal{D}}(F(x), y') \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathcal{C}}(x, G(y)) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(x, G(y'))
\end{array}$$

The commutativity of this diagram follows from the commutativity of the diagram

$$\begin{array}{ccccc}
x & \xrightarrow{e_x} & GF(x) & \longrightarrow & G(y) \\
 & & & \searrow & \downarrow \\
 & & & & G(y')
\end{array}$$

Similarly, naturality in the first variable corresponds to a commutativity of the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{D}}(F(x), y) & \longleftarrow & \mathrm{Hom}_{\mathcal{D}}(F(x'), y) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathcal{C}}(x, G(y)) & \longleftarrow & \mathrm{Hom}_{\mathcal{C}}(x', G(y))
\end{array}$$

given a morphism $x \to x'$ in $\mathcal{C}$.

Proposition 2.3. *An isomorphism $\mathrm{Hom}_{\mathcal{D}}(F(x), y) \cong \mathrm{Hom}_{\mathcal{C}}(x, G(y))$ natural in $x \in \mathcal{C}$ and $y \in \mathcal{D}$ is equivalent to the data of an adjunction $F \dashv G$.*

Proof. We have already shown that an adjunction gives rise to such an isomorphism. Let us conversely start with such a natural isomorphism. Then we get an isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(F(x), F(x)) \cong \mathrm{Hom}_{\mathcal{C}}(x, GF(x)).$$

Let us denote by e_x the image of the identity morphism $F(x) \to F(x)$. To show that e_x is natural consider the commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{D}}(F(x), F(x)) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{C}}(x, GF(x)) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathcal{D}}(F(x), F(x')) & \xrightarrow{\sim} & \mathrm{Hom}(x, GF(x')) \\
\uparrow & & \uparrow \\
\mathrm{Hom}_{\mathcal{D}}(F(x'), F(x')) & \xrightarrow{\sim} & \mathrm{Hom}(x', GF(x'))
\end{array}$$

for a morphism $x \to x'$. Here the top square is commutative due to naturality of the isomorphism in the second variable while the bottom square is commutative due to naturality of the isomorphism in the first variable.

Under the vertical maps both $\mathrm{id}_{F(x)}$ and $\mathrm{id}_{F(x')}$ go to the same morphism $F(x) \to F(x')$ which shows that the diagram

$$\begin{array}{ccc}
x & \longrightarrow & x' \\
\downarrow{\scriptstyle e_x} & & \downarrow{\scriptstyle e_{x'}} \\
GF(x) & \longrightarrow & GF(x')
\end{array}$$

is commutative, i.e. $e\colon \mathrm{id} \Rightarrow GF$ is a natural transformation.

Similarly, under the isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(FG(y), y) \cong \mathrm{Hom}_{\mathcal{C}}(G(y), G(y))$$

the identity morphism $G(y) \to G(y)$ goes to a morphism $\epsilon_y \in \mathrm{Hom}_{\mathcal{D}}(FG(y), y)$ which is also a part of the natural transformation $\epsilon\colon FG \Rightarrow \mathrm{id}$.

Let us now show that the composite

$$F \overset{\mathrm{id}\circ e}{\Rightarrow} FGF \overset{\epsilon\circ\mathrm{id}}{\Rightarrow} F$$

is the identity. For an object $x \in \mathcal{C}$ consider the commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{D}}(FGF(x), F(x)) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{C}}(GF(x), GF(x)) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathcal{D}}(F(x), F(x)) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{C}}(x, GF(x))
\end{array}$$

where the vertical maps come from the precomposition with $e_x\colon x \to GF(x)$. The commutativity of the diagram implies that

$$\epsilon_{F(x)} \circ F(e_x) = \mathrm{id}_{F(x)},$$

which is one of the adjunction axioms. The other axiom is proved in a similar way. □

Theorem 2.4. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. If (G, e, ϵ) exists such that (F, G, e, ϵ) is an adjunction, then (G, e, ϵ) is unique up to unique natural isomorphism. The same result holds true for the left adjoint.*

Proof. Given two right adjoints $G_1, G_2\colon \mathcal{D} \to \mathcal{C}$ we have natural bijections

$$\phi^{(j)}_{x,y} : \mathrm{Hom}_{\mathcal{D}}(F(x), y) \cong \mathrm{Hom}_{\mathcal{C}}(x, G_j(y))$$

for $j = 1, 2$, so

$$\psi_{x,y} = \phi^{(2)}_{x,y} \circ (\phi^{(1)}_{x,y})^{-1} : \mathrm{Hom}_{\mathcal{C}}(x, G_1(y)) \to \mathrm{Hom}_{\mathcal{C}}(x, G_2(y))$$

is a bijection. Put $x = G_1(y)$ and $\eta_y = \psi_{G_1(y),y}(\mathrm{id}_{G_1(y)}) \in \mathrm{Hom}_{\mathcal{C}}(G_1(y), G_2(y))$ for $y \in \mathcal{D}$. By the naturality of $\phi^{(j)}_{x,y}$, this defines a natural transformation $\eta : G_1 \Rightarrow G_2$ taking e_1 to e_2 and ϵ_2 to ϵ_1. Exchanging 1 and 2 and composing shows that η is a natural isomorphism.

The proof that this is the unique natural isomorphism from G_1 to G_2 taking e_1 to e_2 and ϵ_2 to ϵ_1 is then identical to the proof of Proposition 1.11. □

2.3. **Examples.** Many categories that appear in nature are categories of sets equipped with an extra algebraic structure. These have forgetful functors to Set which tend to possess left adjoints.

(1) Consider the forgetful functor $F\colon \mathrm{Vect} \to \mathrm{Set}$. We are going to define a left adjoint Free, where Free(X) is known as the free vector space on a set X. In general, left adjoints to forgetful functors tend to be called "free" functors while right adjoints are called "cofree" functors.

Free(X) is the vector space with basis given by elements of X. One can write it as

$$\mathrm{Free}(X) = \bigoplus_{x \in X} k,$$

where k is the ground field. A linear map $\mathrm{Free}(X) \to V$ is uniquely determined by its value on the basis which gives a map of sets $X \to V$. In other words,

$$\mathrm{Hom}_{\mathrm{Vect}}(\mathrm{Free}(X), V) \cong \mathrm{Hom}_{\mathrm{Set}}(X, F(V))$$

and this isomorphism is natural in the variables X and V. We have proved that Free $\dashv F$.

In the next section we will show that the forgetful functor does not have a right adjoint.

(2) Similarly, the forgetful functor $F\colon \mathrm{Grp} \to \mathrm{Set}$ has a left adjoint $\mathrm{Free}\colon \mathrm{Set} \to \mathrm{Grp}$. The free group Free$(X)$ on a set X has the following description. Denote $Y = X \sqcup X^{-1}$ where $X^{-1} = X$ as a set with elements denoted by the inverse, i.e. if $x \in X$ then $x^{-1} \in X^{-1}$. The free group Free(X) has as its underlying set the set of words on elements Y modulo the cancellations of x and its inverse x^{-1} if they are next to each other. The empty word is the unit and the multiplication is given by concatenation of words.

From the definition it is not difficult to show that

$$\mathrm{Hom}_{\mathrm{Grp}}(\mathrm{Free}(X), G) \cong \mathrm{Hom}_{\mathrm{Set}}(X, F(G)).$$

In practice one just needs to know that there is a left adjoint Free rather than its precise definition.

(3) Consider the forgetful functor $F\colon \mathrm{Ab} \to \mathrm{Grp}$ from abelian groups to groups. Its left adjoint is the abelianization functor $(-)^{ab}\colon \mathrm{Grp} \to \mathrm{Ab}$ where

$$G^{ab} = G/[G, G].$$

(4) Let $\operatorname{Rep} G$ be the category of representations of a finite group G over a field k. We have a functor $\mathrm{triv}\colon \mathrm{Vect} \to \operatorname{Rep} G$ which associates to any vector space the same vector space with the trivial action of G.

Recall that given a representation V we have its subspace of invariants $V^G \hookrightarrow V$ and the quotient space of coinvariants $V \twoheadrightarrow V_G = V/\langle gv - v : v \in V\rangle$. It is easy to construct the following isomorphisms:

$$\mathrm{Hom}_{\operatorname{Rep} G}(\mathrm{triv}(V), W) \cong \mathrm{Hom}_{\mathrm{Vect}}(V, W^G)$$

and

$$\mathrm{Hom}_{\operatorname{Rep} G}(V, \mathrm{triv}(W)) \cong \mathrm{Hom}_{\mathrm{Vect}}(V_G, W).$$

In other words, the functor of invariants is a right adjoint while the functor of coinvariants is a left adjoint. We can write

$$(-)_G \dashv \mathrm{triv} \dashv (-)^G.$$

Now suppose the characteristic of k does not divide $|G|$, the order of the group G. Then we can define a map $V \to V^G$ by

$$v \mapsto \frac{1}{|G|} \sum_{g \in G} g.v$$

which splits the inclusion $V^G \hookrightarrow V$. The map $V \to V^G$ factors through coinvariants and induces the inverse isomorphism to the composite

$$V^G \hookrightarrow V \twoheadrightarrow V_G.$$

Therefore, we have defined a natural isomorphism $(-)_G \cong (-)^G$ of functors. In this case the adjunction between triv and either invariants or coinvariants is an example of an *ambidextrous adjunction* $(-)^G \dashv \mathrm{triv} \dashv (-)^G$.

(5) Let A and B be two algebras and let M be an $A-B$-bimodule, i.e. M is a right A-module together with a commuting left B-module structure. We have a functor $- \otimes_B M$ from right B-modules to right A-modules. Similarly, we have a functor $\underline{\mathrm{Hom}}_A(M, -)$ called the *internal Hom functor* from right A-modules to right B-modules (the B-module structure is given by precomposition with the left B-module structure on M). These are adjoint, i.e. we have a natural isomorphism

$$\mathrm{Hom}_A(N \otimes_B M, L) \cong \mathrm{Hom}_B(N, \underline{\mathrm{Hom}}_A(M, L))$$

for any right B-module N and a right A-module L.

This is known as a *tensor-Hom adjunction* and is immensely useful in (commutative) algebra. For instance, Frobenius reciprocity from group theory is an instance of a tensor-Hom adjunction.

2.4. Definition via initial objects.

Definition 2.5. An *initial object* of a category $\mathcal{C}$ is an object $x \in \mathcal{C}$ such that for every $y \in \mathcal{C}$ the set $\mathrm{Hom}_{\mathcal{C}}(x, y)$ has a unique element.

Dualizing, one gets a definition of final objects:

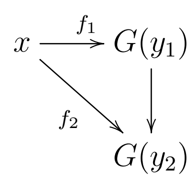
Definition 2.6. A *final object* of a category $\mathcal{C}$ is an object $x \in \mathcal{C}$ such that for every $y \in \mathcal{C}$ the set $\mathrm{Hom}_{\mathcal{C}}(y, x)$ has a unique element.

Notice that initial (and final) objects are unique up to unique isomorphism if they exist, so we can (slightly sloppily) talk about 'the' initial (or final) object.

In the category of sets the initial object is the empty set while in the category of vector spaces the initial object is the zero-dimensional vector space. We will see later that an initial object is an example of a *colimit* in a category, but for now let us use the definition to give an equivalent characterization of adjoints.

Let $F: \mathcal{C} \to \mathcal{D}$ and $G: \mathcal{D} \to \mathcal{C}$ be a pair of functors. Given an object $x \in \mathcal{C}$ consider the following category $(x \Rightarrow G)$:

- Its objects are pairs consisting of an object $y \in \mathcal{D}$ and a morphism $f: x \to G(y)$
- A morphism from (y_1, f_1) to (y_2, f_2) is a morphism $y_1 \to y_2$ making the diagram



  commute.

Remark: The category $(x \Rightarrow G)$ is an example of what is sometimes called a comma category. The category $(F \Rightarrow y)$ for $y \in \mathcal{D}$ is defined similarly.

Theorem 2.7. *The following two structures on the functors F, G are equivalent:*

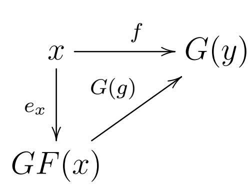
(1) *An adjunction $F \dashv G$,*
(2) *A natural transformation $e: \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ such that $(F(x), e_x)$ is initial in $(x \Rightarrow G)$ for all $x \in \mathcal{C}$.*

Proof.

- **(1)** $\Rightarrow$ **(2)**.

  Suppose $F \dashv G$. Then we want to prove that the unit of the adjunction gives rise to an initial object of $(x \Rightarrow G)$.

  Take an object $(y, f) \in (x \Rightarrow G)$ where $f: x \to G(y)$ and consider a morphism $g: F(x) \to y$ making the diagram



commute. The composite $G(g) \circ e_x = f$ is the image of g under the isomorphism

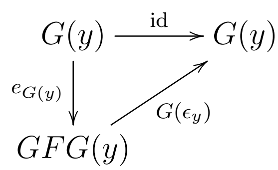
$$\mathrm{Hom}(F(x), y) \cong \mathrm{Hom}(x, G(y)).$$

But then $g \in \mathrm{Hom}(F(x), y)$ is determined uniquely as the preimage of $f \in \mathrm{Hom}(x, G(y))$. In other words, the object (y, f) has a unique morphism from $(F(x), e_x)$.

- $(\mathbf{2}) \Rightarrow (\mathbf{1})$.

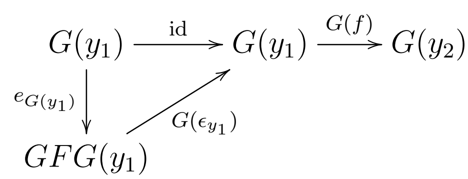
Conversely, suppose $e_x : x \to GF(x)$ defines an initial object $(F(x), e_x)$ in $(x \Rightarrow G)$ for every $x \in \mathcal{C}$. Then we are going to define a counit for the adjunction and show it is unique.

Let $x = G(y)$ and consider the comma category $(G(y) \Rightarrow G)$. Since $(FG(y), e_{G(y)})$ is initial, we have a unique morphism $\epsilon_y : FG(y) \to y$ making the diagram

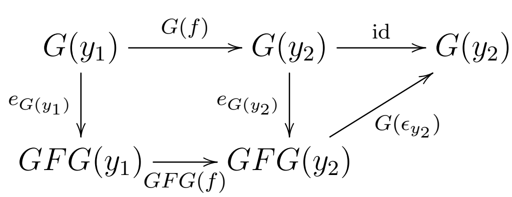


commute.

To show that ϵ is a natural transformation, consider a morphism $f : y_1 \to y_2$. We get commutative diagrams



and



We get two morphisms $FG(y_1) \to y_2$ making the respective diagrams commute: $f \circ \epsilon_{y_1}$ and $\epsilon_{y_2} \circ FG(f)$ and since $(G(y_1), e_{G(y_1)})$ is an initial object of $(G(y_1) \Rightarrow G)$, these must be equal, i.e. ϵ is a natural transformation.

By construction the composite

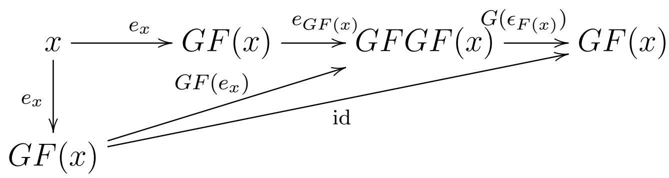
$$G \overset{e \circ \mathrm{id}}{\Rightarrow} GFG \overset{\mathrm{id} \circ \epsilon}{\Rightarrow} G$$

is the identity.

The other axiom states that the composite

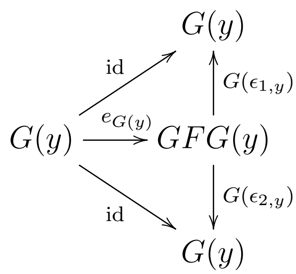
$$F(x) \overset{F(e_x)}{\to} FGF(x) \overset{\epsilon_{F(x)}}{\to} F(x)$$

is the identity. To prove this, consider the commutative diagram



where the triangle commutes due to naturality of e. But then we get two morphisms from the initial object $(F(x), e_x)$ to $(F(x), e_x)$ one of which is given by the identity morphism $F(x) \to F(x)$ and the other one is $\epsilon_{F(x)} \circ F(e_x)$ which are, therefore, equal.

So far we have shown that thus defined counit satisfies the required axioms and we are left to show uniqueness. Suppose ϵ_1 and ϵ_2 are two counits for the adjunction. Then we get a commutative diagram

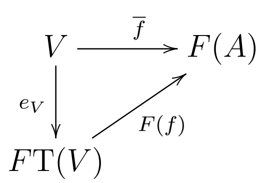


Therefore, $\epsilon_{1,y}$ and $\epsilon_{2,y}$ define morphisms from $(FG(y), e_{G(y)})$ to (y, id) which by initiality must be equal.

□

This theorem is how adjunctions are usually used in practice. For instance, let k be a field and consider the category of vector spaces Vect over k and the category of algebras Alg over k. We have a forgetful functor $F\colon \mathrm{Alg} \to \mathrm{Vect}$ which has a left adjoint $\mathrm{T}\colon \mathrm{Vect} \to \mathrm{Alg}$ known as the tensor algebra functor; in other words, $\mathrm{T}(V)$ is the free algebra on the vector space V.

Suppose A is an algebra. By the previous theorem we get that for any morphism of vector spaces $\overline{f}\colon V \to F(A)$ there is a unique morphism $f\colon \mathrm{T}(V) \to A$ of algebras such that the diagram



commutes. One says that $\mathrm{T}(V)$ is freely generated by V and that maps out of tensor algebras are uniquely determined by the maps on generators.

Let us also state an existence theorem which will be used later to prove a general adjoint functor theorem. The proof is similar to the proof of Theorem 2.6.

Proposition 2.8. *A functor $F\colon \mathcal{C} \to \mathcal{D}$ has a left adjoint iff $(x \Rightarrow F)$ has an initial object for every $x \in \mathcal{D}$. It has a right adjoint iff $(F \Rightarrow x)$ has a final object for every $x \in \mathcal{D}$.*

2.5. **Categorical duality.** Recall that given a category $\mathcal{C}$ we have the opposite category $\mathcal{C}^{op}$ with all morphisms reversed. A functor $\mathcal{C}^{op} \to \mathcal{D}$ is known as a *contravariant* functor from $\mathcal{C}$ to $\mathcal{D}$ while a functor $\mathcal{C} \to \mathcal{D}$ is called a *covariant* functor; these notions make sense only if one has a fixed category $\mathcal{C}$ in mind.

Given a theorem in category theory, one can apply the same theorem to the opposite category and it will still remain true. One often gives definitions or proves statements for some objects in $\mathcal{C}$ and then applies the principle of *categorical duality* to translate the statement to the opposite category.

We will see many examples of this principle later. An easy example would be that a functor $F\colon \mathcal{C} \to \mathcal{D}$ is left adjoint to $G\colon \mathcal{D} \to \mathcal{C}$ iff the functor $F\colon \mathcal{C}^{op} \to \mathcal{D}^{op}$ is right adjoint to $G\colon \mathcal{D}^{op} \to \mathcal{C}^{op}$.

2.6. **Yoneda embedding.** Consider a locally small category $\mathcal{C}$ with an object $x \in \mathcal{C}$. Then we have a functor $\mathrm{Hom}(-, x)\colon \mathcal{C}^{op} \to \mathrm{Set}$. Indeed, a morphism $y_1 \to y_2$ gives a morphism $\mathrm{Hom}(y_2, x) \to \mathrm{Hom}(y_1, x)$ by precomposition. Moreover, the assignment $x \mapsto \mathrm{Hom}(-, x)$ is functorial in the sense that a morphism $x_1 \to x_2$ gives rise to a natural transformation of functors $\mathrm{Hom}(-, x_1) \Rightarrow \mathrm{Hom}(-, x_2)$ by postcomposition. In other words, we have the *Yoneda* functor

$$Y\colon \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set}).$$

Definition 2.9. A functor $F \in \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$ is *representable* if it is in the essential image of the Yoneda functor; that is, it is isomorphic to $Y(c)$ for some $c \in \mathcal{C}$.

The functor category $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$ is sometimes known as the category of *presheaves* on $\mathcal{C}$ (the category of sheaves being a certain subcategory).

Consider an object $x \in \mathcal{C}$ and a presheaf $F \in \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$. A natural transformation between the representable functor $Y(x)$ and F consists of morphisms $\mathrm{Hom}(y, x) \to F(y)$ for every $y \in \mathcal{C}$. In particular, the image of the identity morphism $\mathrm{id}_x \in \mathrm{Hom}(x, x)$ under $\mathrm{Hom}(x, x) \to F(x)$ defines a restriction map $\mathrm{Hom}(Y(x), F) \to F(x)$.

Lemma 2.10 (Yoneda)**.** *Let $x \in \mathcal{C}$ and $F \in \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$. Then the canonical restriction map $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{op},\mathrm{Set})}(Y(x), F) \to F(x)$ is an isomorphism.*

Proof. To show it is an isomorphism, we will construct an inverse $F(x) \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{op},\mathrm{Set})}(Y(x), F)$. Given an element $f \in F(x)$, define a natural transformation $Y(x) \Rightarrow F$ whose components are

$$\mathrm{Hom}(y, x) \to F(y)$$

given by sending $g \in \mathrm{Hom}(y, x)$ to $F(g)(f)$. To show that is is a natural transformation, we have to show that for a morphism $h\colon y_1 \to y_2$ in $\mathcal{C}$ the diagram

$$\begin{array}{ccc} \mathrm{Hom}(y_1, x) & \longrightarrow & F(y_1) \\ {\scriptstyle h}\uparrow & & \uparrow{\scriptstyle F(h)} \\ \mathrm{Hom}(y_2, x) & \longrightarrow & F(y_2) \end{array}$$

is commutative. This immediately follows from the fact that F preserves composition of morphisms.

Clearly, the composite $F(x) \to \mathrm{Hom}(Y(x), F) \to F(x)$ is the identity, so we just have to check that the other composite is the identity.

Let $\eta\colon Y(x) \Rightarrow F$ be a natural transformation with components $\eta_y\colon \mathrm{Hom}(y, x) \to F(y)$. Given a morphism $g \in \mathrm{Hom}(y, x)$ naturality of η implies that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(y,x) & \xrightarrow{\eta_y} & F(y) \\ {\scriptstyle g\circ}\uparrow & & \uparrow{\scriptstyle F(g)} \\ \mathrm{Hom}(x,x) & \xrightarrow{\eta_x} & F(x) \end{array}$$

The image of $\mathrm{id}_x \in \mathrm{Hom}(x, x)$ under the morphism on the left is g, so the commutativity of the diagram implies that $\eta_y(g) = F(g)(\eta_x(\mathrm{id}_x))$ as required. □

Corollary 2.11. *The Yoneda functor is fully faithful.*

Proof. Let $x_1, x_2 \in \mathcal{C}$. We want to show that the morphism

$$\mathrm{Hom}_{\mathcal{C}}(x_1, x_2) \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{op},\mathrm{Set})}(Y(x_1), Y(x_2))$$

is an isomorphism. It is not difficult to check that this morphism is inverse to the canonical restriction

$$\mathrm{Hom}(Y(x_1), Y(x_2)) \to Y(x_2)(x_1) = \mathrm{Hom}_{\mathcal{C}}(x_1, x_2)$$

constructed in the course of the proof of the Yoneda lemma. Therefore, the Yoneda lemma implies that it is an isomorphism. □

Let $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})^{repr} \hookrightarrow \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$ be the subcategory of representable functors, i.e. the essential image of the Yoneda functor. Combining Yoneda lemma with Theorem 1.13 we see that the functor $\mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})^{repr}$ is an equivalence and so there is a functor

$$P\colon \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})^{repr} \to \mathcal{C}$$

such that $Y \dashv P$ forms an adjoint equivalence.

Here is an example where the Yoneda lemma is most clear.

Example. Consider a one-object groupoid $\mathcal{C} = */G$. The presheaf category $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$ can be identified with the category of G-sets $G\mathrm{Set}$. Indeed, every such functor is given by specifying a single set $X \in \mathrm{Set}$ together with a homomorphism $G \to \mathrm{Hom}(X, X)$.

Under the Yoneda embedding $*/G \hookrightarrow G\mathrm{Set}$ the unique object goes to the set G considered as a set with a left G-action. The Yoneda lemma in this case asserts that the morphism $\mathrm{Hom}_{G\mathrm{Set}}(G, X) \to X$ given by evaluating any morphism on the unit element $e \in G$ is an isomorphism. Just like categories generalize groups, the Yoneda lemma generalizes Cayley's theorem from group theory which asserts that every group is a subgroup of the symmetric group on G.

Using categorical duality one also has a contravariant Yoneda embedding

$$\mathcal{C} \to \mathrm{Fun}(\mathcal{C}, \mathrm{Set})^{op}$$

given by sending $x \mapsto \mathrm{Hom}_{\mathcal{C}}(x, -)$. Functors in the essential image of the contravariant Yoneda functor are known as *corepresentable* functors.

2.7. **Adjoints via representable functors.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. Define its *formal right adjoint* G^{formal} to be the functor

$$G^{formal}\colon \mathcal{D} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$$

given by

$$y \mapsto (x \mapsto \mathrm{Hom}_{\mathcal{D}}(Fx, y)).$$

Proposition 2.12. *A right adjoint G to F exists iff $G^{formal}(y)$ is representable for every $y \in \mathcal{D}$.*

Proof.

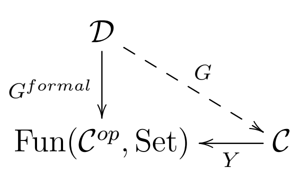
- $\Rightarrow$.
  We have a natural isomorphism $\mathrm{Hom}_{\mathcal{D}}(Fx, y) \cong \mathrm{Hom}_{\mathcal{C}}(x, Gy)$ for every $x \in \mathcal{C}$ and $y \in \mathcal{D}$. Therefore, the functor $G^{formal}(y)$ is representable by the functor $\mathrm{Hom}_{\mathcal{C}}(-, Gy)$.
- $\Leftarrow$.
  We define the right adjoint G to be the composite $G = P \circ G^{formal}$, where $P\colon \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})^{repr} \to \mathcal{C}$ is the inverse to the Yoneda embedding Y.
  To show that $F \dashv G$, we have to construct an isomorphism $\mathrm{Hom}_{\mathcal{D}}(Fx, y) \cong \mathrm{Hom}_{\mathcal{C}}(x, Gy)$ natural in x and y. It is constructed in the following way:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(x, Gy) &= \mathrm{Hom}_{\mathcal{C}}(x, PG^{formal}y)\\ &\cong \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{op},\mathrm{Set})}(Yx, G^{formal}y)\\ &\cong G^{formal}(y)(x)\\ &= \mathrm{Hom}_{\mathcal{D}}(Fx, y),\end{aligned}$$

  where in the second line we have used the adjoint equivalence $Y \dashv P$ and in the third line we applied the Yoneda lemma.

□

In other words, adjoints can be thought of as factorizations of the diagram



More precisely, a right adjoint consists of a functor $G\colon \mathcal{D} \to \mathcal{C}$ and a natural isomorphism $YG \cong G^{formal}$. Using Yoneda lemma one can give an easier proof of Theorem 2.4.

Indeed, suppose G_1 and G_2 are two right adjoints. Then we have isomorphisms

$$YG_1 \cong G^{formal} \cong YG_2.$$

Since the Yoneda embedding is fully faithful, a natural isomorphism $YG_1 \cong YG_2$ is equivalently a natural isomorphism $G_1 \cong G_2$.

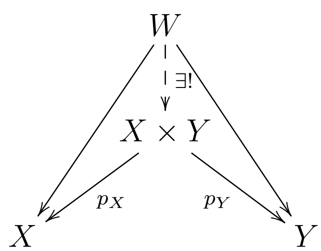
3. Limits and colimits

3.1. **Examples.** We now come to another important construction in category theory: limits and colimits. These are certain constructions defined by universal properties examples of which include many known constructions such as amalgamated products of groups, direct sums of vector spaces, tensor products of algebras and so on. Before we give a definition of a general limit, we will define particular instances of this phenomenon.

Recall the notion of a *final* object: this is an object $* \in \mathcal{C}$ such that there is a unique morphism $x \to *$ for every $x \in \mathcal{C}$. It is immediate that a final object is unique up to a unique isomorphism if it exists: if $*'$ is another final object there is a unique morphism $* \to *'$ which is necessarily an isomorphism.

Another example of a limit is a product.

Definition 3.1. Let $X, Y \in \mathcal{C}$. We say that a collection $(X \times Y, p_X, p_Y)$ where $X \times Y \in \mathcal{C}$, $p_X\colon X \times Y \to X$ and $p_Y\colon X \times Y \to Y$ is a *product* if for every object W together with maps $W \to X$ and $W \to Y$ there is a unique morphism $W \to X \times Y$ making the diagram



commute.

Here are some examples.

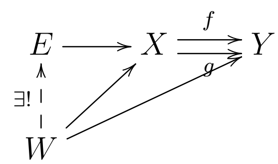
- The product in Set is the Cartesian product of sets: $X \times Y$ has projection maps to X and Y and whenever Z maps to X and Y one has a morphism to $X \times Y$ given by the product of the maps. To show that it is unique, note that an arbitrary morphism $Z \to X \times Y$ can be written uniquely as $z \mapsto (f_X(z), f_Y(z))$.
- One can similarly define products in Grp to be the Cartesian product of groups $G \times H$ with the multiplication given pointwise: $(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2)$.
- The product in abelian groups Ab is given by the same formula (note that the product of two abelian groups is automatically abelian). If G and H are abelian groups, the product is also denoted by $G \oplus H$.
- Product in Vect is given by the direct sum of vector spaces $V \oplus W$.

One can also generalize the product from two objects to objects parametrized by a set I. In this case the product $\prod_{i \in I} X_i$ is defined to be the object with projections to X_i satisfying a universal property as before.

Remark. A collection of objects of $\mathcal{C}$ parametrized by a set I is the same as a functor $I \to \mathcal{C}$ where I is considered as a discrete category. This will motivate a definition of a general limit.

Yet another example of a limit is an equalizer.

Definition 3.2. Let $X, Y \in \mathcal{C}$ together with two maps $f, g\colon X \to Y$. An *equalizer* of $X \rightrightarrows Y$ is an object $E \in \mathcal{C}$ together with a map $E \to X$ such that the two composites $E \to Y$ are equal satisfying the following universal property: for every W with compatible maps to X, Y there is a unique map $W \to E$ making the diagram



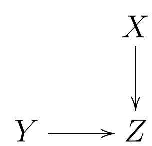
commute.

One denotes an equalizer by $\mathrm{eq}(X \rightrightarrows Y)$.

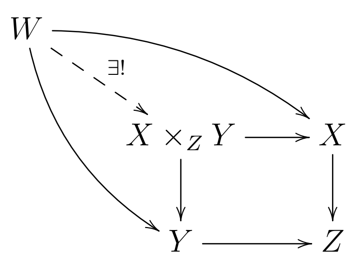
- Let us work out equalizers in the category of sets. Consider two morphisms $f, g\colon X \to Y$ and a morphism $h\colon W \to X$ such that $f \circ h = g \circ h$. Clearly, the map $W \to X$ factors through the subset $E \subset X$ defined by $E = \{x \in X | f(x) = g(x)\}$ which is the equalizer.
- Suppose G, H are abelian groups and one has maps $f, g\colon G \to H$. Then the equalizer $\mathrm{eq}(G \rightrightarrows H)$ can be defined to be the kernel $\ker(f - g\colon G \to H)$. More generally, this defines the equalizer in the category of R-modules for any ring R.

The final example of a limit we will give is that of a fibre product.

Definition 3.3. Consider a diagram

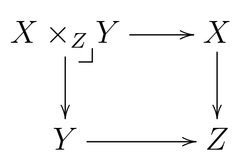


A *fibre product* $X \times_Z Y$ is an object with morphisms to X and Y such that the two composites $X \times_Z Y \to Z$ are equal, and satisfying the following universal property: for every object W with maps to X and Y such that the two composites $W \to Z$ are equal there is a unique morphism $W \to X \times_Z Y$ making the diagram



commute.

A fibre product is also known as a *pullback*. We say that the squares of the form



are *Cartesian*. This is denoted by a little corner symbol.

fibre products can be written as combinations of products and equalizers:

$$\mathrm{eq}(X \times Y \rightrightarrows Z) \cong X \times_Z Y.$$

Indeed, a morphism $W \to X \times Y$ is the same as a pair of morphisms $W \to X, Y$ and the equalizer condition implies that the two composites $W \to Z$ are equal.

Conversely, if $* \in \mathcal{C}$ is the final object we have

$$X \times Y \cong X \times_* Y.$$

An example of a fibre product in sets is the inverse image: given a morphism $f\colon X \to Y$ and an element $y \in Y$ which we represent as a morphism $* \to Y$ the fibre product $X \times_Y *$ is the inverse image $f^{-1}(y)$.

3.2. **Limits.** Perhaps, the reader has already noticed a general pattern. A limit involves a diagram of certain shapes and it's an object satisfying a universal property. Let us make this precise.

Definition 3.4. Let $F\colon I \to \mathcal{C}$ be a functor from a category I. A *limit* of F is an object $\lim_I F \in \mathcal{C}$ together with maps $f_i\colon \lim_I F \to F(i)$ for every $i \in I$ such that for every morphism $g\colon i \to j$ in I one has $F(g) \circ f_i = f_j$ and it is a universal such object; that is, for every object W with compatible projections $W \to F(i)$ one has a unique morphism $W \to \lim_I F$.

A functor $F\colon I \to \mathcal{C}$ is called a *diagram of shape I*.

Here is a reformulation. Given a diagram F a *cone* on F is an object $x \in \mathcal{C}$ together with morphisms $x \to F(i)$ which are compatible as in the definition of limits. Cones form a category and a limit is a final object in that category. The dual notion to a cone is called a *cocone*.

- If I is the empty category we recover the notion of a final object.
- If I is a discrete category, we recover the notion of a product indexed by I.
- If I is the category with two objects and two non-identity morphisms as in

$$* \rightrightarrows *,$$

  then we recover equalizers.
- If $I = *$, the limit is simply the value of the functor on the unique object.
- If I is the poset of natural numbers $\mathbb{N}$, limits of shape I^{op} are known as inverse limits.

Proposition 3.5. *A limit, if it exists, is unique up to a unique isomorphism.*

Proof. Suppose $\lim_I F$ and $\lim'_I F$ are two limits of the functor $F\colon I \to \mathcal{C}$. By the universal property we have unique morphisms $\lim_I F \to \lim'_I F$ and $\lim'_I F \to \lim_I F$ making the natural diagrams commute. Clearly, they compose to the identity as there is a unique morphism $\lim_I F \to \lim_I F$ making the natural diagrams commute. □

Let us give another point of view on limits. A morphism $W \to \lim_I F$ is the same as a collection of morphisms $p_i\colon W \to F(i)$ such that for every morphism $g\colon i \to j$ one has $F(g) \circ p_i = p_j$.

Let us introduce the diagonal functor $\Delta\colon \mathcal{C} \to \mathrm{Fun}(I, \mathcal{C})$ given by sending $x \in \mathcal{C}$ to the constant functor $i \mapsto x$ (and morphisms sent to composition with themselves). Then a morphism $W \to \lim_I F$ is the same as a natural transformation $\Delta(W) \Rightarrow F$. In other words, we have a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Fun}(I,\mathcal{C})}(\Delta W, F) \cong \mathrm{Hom}_{\mathcal{C}}(W, \lim_I F).$$

Theorem 3.6. *Suppose $\mathcal{C}$ has all limits of shape I. Then the limit $\lim_I \colon \mathrm{Fun}(I,\mathcal{C}) \to \mathcal{C}$ is a right adjoint to the diagonal functor $\Delta\colon \mathcal{C} \to \mathrm{Fun}(I,\mathcal{C})$.*

Suppose I and J are two categories and consider a diagram $F\colon I \times J \to \mathcal{C}$. Then we can try to compare $\lim_I \lim_J F$ and $\lim_J \lim_I F$.

Theorem 3.7. *Suppose $\mathcal{C}$ has limits for diagrams of shapes I and J. Then it has limits for diagrams of shape $I \times J$ and*

$$\lim_{I\times J} F \cong \lim_I \lim_J F \cong \lim_J \lim_I F$$

for any diagram $F\colon I \times J \to \mathcal{C}$.

Proof. Let $F\colon I \times J \to \mathcal{C}$ be a diagram. In the proof we will freely reinterpret it as a functor $I \to \mathrm{Fun}(J,\mathcal{C})$.

Let $\Delta_I\colon \mathcal{C} \to \mathrm{Fun}(I,\mathcal{C})$, $\Delta_J\colon \mathrm{Fun}(I,\mathcal{C}) \to \mathrm{Fun}(J,\mathrm{Fun}(I,\mathcal{C}))$ and $\Delta_{I\times J}\colon \mathcal{C} \to \mathrm{Fun}(I\times J,\mathcal{C})$ be the diagonal functors.

Take $x \in \mathcal{C}$ and consider the sequence of natural isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(x, \lim_I \lim_J F) &\cong \mathrm{Hom}_{\mathrm{Fun}(I,\mathcal{C})}(\Delta_I(x), \lim_J F)\\ &\cong \mathrm{Hom}_{\mathrm{Fun}(J,\mathrm{Fun}(I,\mathcal{C}))}(\Delta_J(\Delta_I(x)), F)\\ &\cong \mathrm{Hom}_{\mathrm{Fun}(I\times J,\mathcal{C})}(\Delta_{I\times J}(x), F).\end{aligned}$$

This tells us that $\lim_I \lim_J F$ has the properties required to be the limit $\lim_{I\times J} F$; in other words, since these properties define the limit up to unique isomorphism, the limit $\lim_{I\times J} F$ exists and is isomorphic to $\lim_I \lim_J F$. But we also get an isomorphism

$$\lim_J \lim_I F \cong \lim_{I\times J} F.$$

Therefore,

$$\lim_I \lim_J F \cong \lim_J \lim_I F.$$

□

In other words, limits commute and we can interchange them. This is a kind of Fubini theorem for limits.

3.3. **Constructing limits.** We say that a category $\mathcal{C}$ *has limits* (or it is *complete*) if for every (small) category I and every functor $F\colon I \to \mathcal{C}$ there is a limit $\lim_I F$. We say that a category *has finite limits* if it has limits for all shapes I where I is a finite category (i.e. a category with finitely many objects and finitely many morphisms).

Let us begin by proving that Set has all limits. Consider a diagram $F\colon I \to \mathrm{Set}$. Then we must have

$$\lim_I F \cong \mathrm{Hom}_{\mathrm{Set}}(*, \lim_I F) \cong \mathrm{Hom}_{\mathrm{Fun}(I,\mathrm{Set})}(\Delta(*), F).$$

The latter set is the set of morphisms $* \to F(i)$, i.e. elements $x_i \in F(i)$, such that for every morphism $f\colon i \to j$ one has $f(x_i) = x_j$. Note that we have constructed this limit as a subset of the product satisfying some equations.

More generally, we have the following theorem.

Theorem 3.8.

- *A category $\mathcal{C}$ has limits iff it has products and equalizers.*
- *A category $\mathcal{C}$ has finite limits iff it has binary products, final object and equalizers.*

Proof. Clearly, if a category has limits, then it has products and equalizers.

Conversely, suppose the category $\mathcal{C}$ has products and equalizers and consider a diagram $F\colon I \to \mathcal{C}$. We want to repeat the construction of limits in Set in $\mathcal{C}$. For every morphism $f\colon i \to j$ we define a pair of morphisms

$$\prod_{k\in I} F(k) \to F(j)$$

as follows. One of the morphisms is simply the projection $\prod_{k\in I} F(k) \to F(j)$ on the component $k = j$. The other map is given by the composite

$$\prod_{k\in I} F(k) \to F(i) \stackrel{F(f)}{\to} F(j).$$

By the universal property of the product we get maps

$$\prod_{k\in I} F(k) \rightrightarrows \prod_{(i\to j)\in \mathrm{Fun}([1],I)} F(j).$$

Here $\mathrm{Fun}([1], I)$ is the category of arrows in I, i.e. pairs of objects $i, j \in I$ together with a morphism $i \to j$.

We claim that the equalizer E of this diagram is the limit of F. Indeed, the universal property of the equalizer is that for any morphism $W \to \prod_{k\in I} F(k)$, i.e. for any collection of morphisms $W \to F(k)$, such that the composite $W \to F(i) \stackrel{F(f)}{\to} F(j)$ coincides with $W \to F(j)$ for every $f\colon i \to j$ there is a unique morphism $W \to E$. But this is exactly the universal property of the limit.

The same proof shows that a category has finite limits iff it has finite products and equalizers. But any finite product is a final object (if the indexing set is empty), the identity functor (if the indexing set has a single element) or can be constructed as an iterated application of binary products. □

Let us give another way to construct limits. We want to generalize the example of inverse limits to so-called cofiltered limits.

Definition 3.9. A category I is *cofiltered* if it satisfies the following three properties:

- It is non-empty.
- For every pair of objects $i, j \in I$ there is $k \in I$ together with morphisms $k \to i$ and $k \to j$.
- For every pair of objects $i, j \in I$ and a pair of morphisms $u, v\colon i \to j$ there is an object $k \in I$ together with a morphism $k \to i$ such that the diagram $k \to i \rightrightarrows j$ commutes.

Equivalently, a cofiltered category is one which has cones for every finite diagram.

We say that a category I is *filtered* if I^{op} is cofiltered. That is, a filtered category is one which has cocones for every finite diagram.

For instance, the category $\mathbb{N}$ is filtered while $\mathbb{N}^{op}$ cofiltered. We call limits over cofiltered diagrams *cofiltered limits.*

Theorem 3.10. *A category $\mathcal{C}$ has limits iff it has cofiltered limits and finite limits.*

Proof. Suppose $\mathcal{C}$ has cofiltered and finite limits. Let us prove that $\mathcal{C}$ has products.

Indeed, consider a set I and a functor $F\colon I \to \mathcal{C}$. Consider the category I^+ of finite subsets of I with morphisms given by inclusion. Then $(I^+)^{op}$ is cofiltered. We define a functor $F^+\colon (I^+)^{op} \to \mathcal{C}$ by sending a finite subset $J \subset I$ to $\lim_J F|_J$. This is well-defined since we assume the category has finite limits. It is not difficult to show then that the limit $\lim_{(I^+)^{op}} F^+$ gives the product $\prod_I F$. □

3.4. **Colimits.** A *colimit* is a notion dual to that of a limit: given a diagram $F\colon I \to \mathcal{C}$ we define

$$\mathrm{colim}_I F = \lim_{I^{op}} F,$$

where F is identified with $F\colon I^{op} \to \mathcal{C}^{op}$.

Let us now dualize the definitions of limits to understand the colimits better.

Definition 3.11. A *colimit* of a diagram $F\colon I \to \mathcal{C}$ is an object $\mathrm{colim}_I F \in \mathcal{C}$ together with maps $f_i\colon F(i) \to \mathrm{colim}_I F$ for every $i \in I$ such that for every morphism $g\colon i \to j$ in I one has $f_j \circ F(g) = f_i$ and it is a universal such object; that is, for every object W with compatible maps $F(i) \to W$ one has a unique morphism $\mathrm{colim}_I F \to W$.

This definition can be succinctly summarized by saying that a colimit is an initial cocone on F (cocones are dual to cones).

One can also define it as a left adjoint to the diagonal functor Δ:

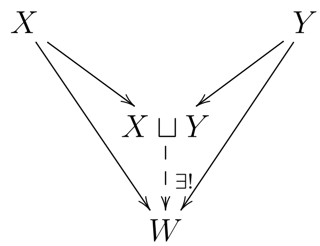
Theorem 3.12. *Colimit* $\mathrm{colim}_I\colon \mathrm{Fun}(I, \mathcal{C}) \to \mathcal{C}$ *is left adjoint to the diagonal functor* $\Delta\colon \mathcal{C} \to \mathrm{Fun}(I, \mathcal{C})$ *sending an object of $\mathcal{C}$ to the constant functor.*

We have the following basic shapes of colimits:

- Colimit over $I = \varnothing$ is the same as an initial object.
- Colimit over $I = *$ is the value of the functor on the unique object in I.
- Colimit over $I = * \sqcup *$ is a *coproduct.*
- Colimit over $I = * \rightrightarrows *$ is a *coequalizer.*
- Colimit over $* \leftarrow * \rightarrow *$ is a *pushout.*
- Colimit over the poset $\mathbb{N}$ is an *inductive limit.*

Let us now give examples of these concepts.

Definition 3.13. A *coproduct* of $X, Y \in \mathcal{C}$ is an object $X \sqcup Y \in \mathcal{C}$ together with morphisms $X, Y \to X \sqcup Y$ such that for every object W and morphisms $X, Y \to W$ one has a unique morphism $X \sqcup Y \to W$ making the diagram



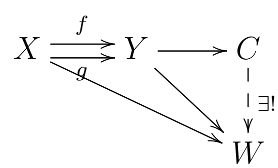
commute.

Here are some examples:

- In Set the coproduct is simply the disjoint union. If one has morphisms $X, Y \to W$ there is a unique morphism $X \sqcup Y \to W$ which has the right components.
- In the category of vector spaces the coproduct is the direct sum of vector spaces $V \oplus W$.
- In the category of groups the coproduct is the free product of groups $G * H$.
- In the category of abelian groups the coproduct is the direct product of groups $G \times H \equiv G \oplus H$. Note that it is different from the one in the category of groups!

Next, let's discuss coequalizers.

Definition 3.14. Let $X, Y \in \mathcal{C}$ together with two maps $f, g\colon X \to Y$. A *coequalizer* of $X \rightrightarrows Y$ is an object $C \in \mathcal{C}$ together with a map $Y \to C$ such that the two composites $X \to C$ are equal satisfying the following universal property: for every W with compatible maps from X, Y there is a unique map $C \to W$ making the diagram

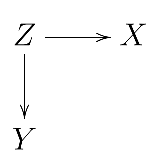


commute.

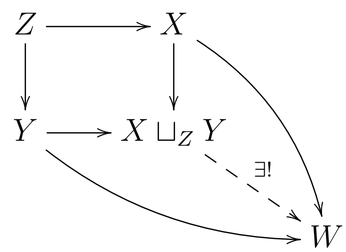
For a pair of morphisms $f, g\colon X \to Y$ we denote the coequalizer by $\mathrm{coeq}(X \rightrightarrows Y)$. Let us work it out in the category of sets. This should be a set C with a map from Y such that the two composites $X \overset{f,g}{\to} Y \to C$ are the same. We define $C = Y/\sim$ where we identify $f(x) \sim g(x)$ for every $x \in X$. Suppose we have a map $Y \to W$ such that the two composites $X \overset{f,g}{\to} Y \to W$ are the same. Therefore, we can factor the map $Y \to W$ as $Y \to C \to W$. Since $Y \to C$ is surjective, such a factorization is unique.

Next we discuss pushouts.

Definition 3.15. Consider a diagram

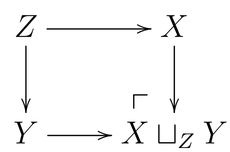


A *pushout* $X \sqcup_Z Y$ is an object with maps from X and Y such that the two composites $Z \to X \sqcup_Z Y$ are equal, satisfying the following universal property: for every object W with maps from X and Y such that the two composites $Z \to W$ are equal there is a unique morphism $X \sqcup_Z Y \to$ making the diagram



commute.

We call commutative squares of the form



coCartesian squares and denote this by a little corner symbol similarly to Cartesian squares.

As for pullbacks, we can define them as a combination of coproducts and coequalizers:

$$X \sqcup_Z Y \cong \operatorname{coeq}(Z \rightrightarrows X \sqcup Y),$$

where the morphisms $Z \to X \sqcup Y$ are the composites $Z \to X \to X \sqcup Y$ and $Z \to Y \to X \sqcup Y$.

Finally, let us discuss an example of a filtered colimit which is an *inductive limit* (also known as a direct limit). These are colimits over the poset $\mathbb{N}$. So, consider a sequence of sets $X_0, X_1, X_2, ...$ together with morphisms $f_i \colon X_i \to X_{i+1}$. An inductive limit

$$\operatorname{colim}_{n\in\mathbb{N}} X_n$$

can be described as the disjoint union of all X_n modulo the following equivalence relation: $x_i \sim f_i(x_i)$ for every $x_i \in X_i$.

We can dualize Theorems 3.8 and 3.10 for colimits.

Theorem 3.16.

- *A category $\mathcal{C}$ has colimits iff it has coproducts and coequalizers.*
- *A category $\mathcal{C}$ has colimits iff it has filtered colimits and finite colimits.*
- *A category $\mathcal{C}$ has finite colimits iff it has binary coproducts, initial object and coequalizers.*

In particular, since Set has coproducts, coequalizers, products and equalizers we see that it is both complete and cocomplete, i.e. it has all limits and all colimits.

We also have a dual theorem to Theorem 3.7:

Theorem 3.17. *Suppose $\mathcal{C}$ has colimits over diagrams of shape I and J. Then it has colimits over diagrams of shape $I \times J$ and*

$$\operatorname{colim}_{I\times J} F \cong \operatorname{colim}_I \operatorname{colim}_J F \cong \operatorname{colim}_J \operatorname{colim}_I F$$

for every diagram $F \colon I \times J \to \mathcal{C}$.

Note that, however, limits and colimits do not interchange in general: if X, Y, Z, W are sets, the sets $(X \sqcup Y) \times (Z \sqcup W)$ and $(X \times Z) \sqcup (Y \times W)$ are not isomorphic in general since

$$(X \sqcup Y) \times (Z \sqcup W) \cong (X \times Z) \sqcup (X \times W) \sqcup (Y \times Z) \sqcup (Y \times W).$$

Nevertheless, we have the following theorem about filtered colimits and finite limits in sets which we state without proof.

Theorem 3.18. *Filtered colimits and finite limits commute in* Set.

3.5. **Epimorphisms and monomorphisms.** In the category of sets we have a notion of an injective map: a map $f\colon X \to Y$ is *injective* if for all elements $x, y \in X$ the equality $f(x) = f(y)$ implies $x = y$. We can reformulate it in a slightly more categorical way: a map $f\colon X \to Y$ is injective if for all maps $x, y\colon * \to X$ the equality $f \circ x = f \circ y$ implies $x = y$. Note that this definition is equivalent to the one where we replace $*$ by any other set.

Definition 3.19. A morphism $f\colon X \to Y$ in a locally small category $\mathcal{C}$ is a *monomorphism* if the induced morphism $\mathrm{Hom}_{\mathcal{C}}(Z, X) \to \mathrm{Hom}_{\mathcal{C}}(Z, Y)$ is injective for every object $Z \in \mathcal{C}$. That is, an equality $f \circ g_1 = f \circ g_2$ for two morphisms $g_i\colon Z \to X$ implies that $g_1 = g_2$.

Dually, we have epimorphisms.

Definition 3.20. A morphism $f\colon X \to Y$ is an *epimorphism* if the induced morphism $\mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$ is injective for all Z.

We have already mentioned that monomorphisms and epimorphisms in Set are just injective or surjective morphisms. Here is a characterization of a class of monomorphisms and epimorphisms in concrete categories.

Proposition 3.21. *Suppose $\mathcal{C}$ is a category with a faithful functor F to* Set. *Then a morphism $f\colon x \to y$ is a monomorphism or epimorphism if it is so in* Set.

Proof. Let us assume that $F(f)\colon F(x) \to F(y)$ is a monomorphism. Take an object $z \in \mathcal{C}$ and consider a diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(z, x) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(z, y) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathrm{Set}}(Fz, Fx) & \longrightarrow & \mathrm{Hom}_{\mathrm{Set}}(Fz, Fy) \end{array}$$

Since $F(f)$ is a monomorphism, the bottom map is injective. Since F is faithful, the vertical maps are injective. Therefore, the top map is injective as well, i.e. f is a monomorphism.

The statement about epimorphisms is proved by dualizing. □

Many categories arising in practice are categories of sets equipped with some extra structure. In particular, all such categories have faithful functors to Set, so the theorem applies.

In the categories of groups, abelian groups and vector spaces monomorphisms and epimorphisms are exactly injective and surjective morphisms. However, a morphism of monoids $\mathbb{N} \to \mathbb{Z}$ is an epimorphism even though it is not surjective: a morphism $\mathbb{Z} \to M$ for some monoid M is an invertible element $m \in M$ while a morphism $\mathbb{N} \to M$ is any element of M. Clearly, if two morphisms $\mathbb{Z} \to M$ give the same composites $\mathbb{N} \to \mathbb{Z} \to M$ they have to be equal.

Definition 3.22. We say that a morphism $X \to Y$ is a *regular monomorphism* if it exhibits X as an equalizer of some diagram $Y \rightrightarrows Y'$.

Dually we have a notion of a regular epimorphism.

Definition 3.23. A morphism $X \to Y$ is a *regular epimorphism* if it exhibits Y as a coequalizer of some diagram $X' \rightrightarrows X$.

Proposition 3.24. *A regular monomorphism is a monomorphism. Dually, a regular epimorphism is an epimorphism.*

Proof. Suppose

$$X \xrightarrow{h} Y \underset{f_2}{\overset{f_1}{\rightrightarrows}} Y'$$

is an equalizer diagram. We want to prove that $h\colon X \to Y$ is a monomorphism.

Indeed, consider two morphisms $g_1, g_2\colon Z \to X$ such that their composites $Z \to X \to Y$ with h are equal. By the definition of an equalizer we have

$$f_1 \circ h = f_2 \circ h$$

which implies that

$$f_1 \circ h \circ g_1 = f_1 \circ h \circ g_2 = f_2 \circ h \circ g_2 = f_2 \circ h \circ g_1.$$

Therefore, by the universal property of the equalizer there is a unique morphism $g\colon Z \to X$ such that $h \circ g = h \circ g_i$ for $i = 1, 2$. In other words, $g_1 = g_2$.

The statement about epimorphisms is proved by duality. □

4. Adjoint functors and limits

4.1. **Functors and limits.** Suppose $F\colon \mathcal{C} \to \mathcal{D}$ is a functor and consider a diagram $J\colon I \to \mathcal{C}$. Clearly, F sends cones on J to cones on $F \circ J$. Therefore, we can ask what happens to special cones, namely limits, under this functor.

Definition 4.1. We say that F *preserves limits* if for any diagram $I \to \mathcal{C}$ the functor F sends a universal cone to a universal cone.

Definition 4.2. We say that F *reflects limits* if for any diagram $J\colon I \to \mathcal{C}$ and any cone on J which is sent to a universal cone is universal.

Dually, we talk about preservation and reflection of colimits.

Here is a more explicit way to write the condition of preservation of limits. Suppose $J\colon I \to \mathcal{C}$ is a diagram in $\mathcal{C}$ which has a limit. Then $F \circ J$ has a limit in $\mathcal{D}$ and the canonical morphism

$$F(\lim_I J) \to \lim_I F \circ J$$

is an isomorphism. The morphism is constructed as follows: we have compatible projection maps $\lim_I J \to J(i)$ for every $i \in I$ which are sent to compatible projection maps $F(\lim_I J) \to (F \circ J)(i)$, so by universal property we have a unique morphism

$$F(\lim_I J) \to \lim_I F \circ J.$$

Here is a basic example of a functor preserving limits.

Theorem 4.3. *Suppose $\mathcal{C}$ is a locally small category. Then the* Hom *functor*

$$\mathrm{Hom}_{\mathcal{C}}(x, -)\colon \mathcal{C} \to \mathrm{Set}$$

preserves limits for any $x \in \mathcal{C}$.

Proof. Suppose $J\colon I \to \mathcal{C}$ is a diagram with a limit in $\mathcal{C}$ and consider an object $x \in \mathcal{C}$.

Let us start with the following observation:

$$\mathrm{Hom}_{\mathrm{Fun}(I,\mathcal{C})}(\Delta(x), J) \cong \mathrm{Hom}_{\mathrm{Fun}(I,\mathrm{Set})}(\Delta(*), \mathrm{Hom}_{\mathcal{C}}(x, J)).$$

Indeed, a morphism on the left is a compatible family of morphisms $x \to J(i)$ while a morphism on the right is a compatible family of morphisms $* \to \mathrm{Hom}_{\mathcal{C}}(x, J(i))$.

We get the following chain of isomorphisms natural in x:

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Set}}(*, \mathrm{Hom}_{\mathcal{C}}(x, \lim_I J)) &\cong \mathrm{Hom}_{\mathcal{C}}(x, \lim_I J)\\ &\cong \mathrm{Hom}_{\mathrm{Fun}(I,\mathcal{C})}(\Delta(x), J)\\ &\cong \mathrm{Hom}_{\mathrm{Fun}(I,\mathrm{Set})}(\Delta(*), \mathrm{Hom}_{\mathcal{C}}(x, J)).\end{aligned}$$

Therefore, $\mathrm{Hom}_{\mathcal{C}}(x, J)$ has a limit which is isomorphic to $\mathrm{Hom}_{\mathcal{C}}(x, \lim_I J)$. □

By duality the functor

$$\mathrm{Hom}_{\mathcal{C}}(-, x)\colon \mathcal{C}^{op} \to \mathrm{Set}$$

takes colimits in $\mathcal{C}$ to limits, i.e. for any diagram $J\colon I \to \mathcal{C}$ with a colimit we have

$$\mathrm{Hom}(\mathrm{colim}_I J, x) \cong \lim_I \mathrm{Hom}(J, x).$$

Proposition 4.4. *Suppose $F\colon \mathcal{C} \leftrightarrows \mathcal{D} : G$ is an adjunction with $F \dashv G$. Then G preserves limits while F preserves colimits.*

Proof. We will prove the statement for the right adjoint, the other statement is proved by duality.

Suppose $J\colon I \to \mathcal{D}$ is a diagram with a limit in $\mathcal{D}$. Then for any object $x \in \mathcal{C}$ we have a chain of isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(x, G \lim_I J) &\cong \mathrm{Hom}_{\mathcal{D}}(Fx, \lim_I J)\\ &\cong \lim_I \mathrm{Hom}_{\mathcal{D}}(Fx, J)\\ &\cong \lim_I \mathrm{Hom}_{\mathcal{C}}(x, G \circ J)\\ &\cong \mathrm{Hom}_{\mathcal{C}}(x, \lim_I G \circ J).\end{aligned}$$

□

This proposition gives a necessary condition for the existence of adjoints: if a right adjoint exists, the functor has to preserve colimits. The forgetful functors Grp → Set and Vect → Set do not preserve colimits (e.g. they do not send initial objects to initial objects), so they do not have right adjoints.

Let us give more examples. Clearly, the forgetful functor Grp → Set preserves and reflects limits. But more is true. Consider two groups G_1 and G_2. We claim that their product has a unique group structure such that the projections $G_1 \times G_2 \to G_i$ are homomorphisms. Indeed, the product of (g_1, g_2) and (h_1, h_2) has to be $(g_1h_1, ...)$ since the projection to G_1 is a homomorphism. The same argument for the second factor shows that $(g_1, g_2) \cdot (h_1, h_2) = (g_1 h_1, g_2 h_2)$. In this case we say that the forgetful functor Grp $\to$ Set creates limits.

Definition 4.5. A functor $F\colon \mathcal{C} \to \mathcal{D}$ *creates limits* if for any diagram $J\colon I \to \mathcal{C}$ and any universal cone C on $F \circ J$ its preimage in cones on J is nonempty and consists of universal cones.

In other words, for any final cone on $F \circ J$ there is a unique cone on J up to a unique isomorphism which maps to the given cone and that cone on J is final. The following statement explains the terminology of "creation" of limits.

Proposition 4.6. *Suppose $F\colon \mathcal{C} \to \mathcal{D}$ is a functor. Suppose that limits of shape I exist in $\mathcal{D}$ and F creates them. Then limits of shape I exist in $\mathcal{C}$ and F preserves and reflects them.*

Proof. For any diagram $J\colon I \to \mathcal{C}$ there is a universal cone in $\mathcal{D}$ on $F \circ J$. Since F creates limits of shape I, there is a universal cone on J in $\mathcal{C}$, so $\mathcal{C}$ has limits of shape I. Clearly, F reflects limits of shape I.

To show that F preserves limits of shape I, suppose C is a universal cone on J and D is a universal cone on $F \circ J$. Since F creates limits of shape I, there is a universal cone D' on J which maps to D. But any universal cones are isomorphic, i.e. $C \cong D'$ which implies that $F(C) \cong D$, i.e. $F(C)$ is universal. □

4.2. **Limits in functor categories.** Suppose $\mathcal{C}$ and $\mathcal{D}$ are two categories such that $\mathcal{D}$ has certain limits or colimits. In this section we will prove that the functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ has those limits or colimits and they are computed pointwise.

In the course we mainly work with locally small categories. The functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is, however, not locally small even if $\mathcal{C}$ and $\mathcal{D}$ are. However, if we assume that $\mathcal{C}$ is small and $\mathcal{D}$ is locally small, the functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is still locally small.

Theorem 4.7. *Suppose $\mathcal{D}$ has limits of shape I and denote by $\lim_I\colon \mathrm{Fun}(I, \mathcal{D}) \to \mathcal{D}$ the limit functor. Then the composite*

$$\mathrm{Fun}(I, \mathrm{Fun}(\mathcal{C}, \mathcal{D})) \cong \mathrm{Fun}(\mathcal{C}, \mathrm{Fun}(I, \mathcal{D})) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

is the limit functor. In other words, limits in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ are computed pointwise.

Dually, if $\mathcal{D}$ has colimits of shape I, colimits in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ are computed pointwise.

Proof. Consider the adjunction $\lim_I \vdash \Delta$ between categories $\mathrm{Fun}(I, \mathcal{D}) \leftrightarrows \mathcal{D}$. We claim that it induces an adjunction

$$\mathrm{Fun}(\mathcal{C}, \mathrm{Fun}(I, \mathcal{D})) \leftrightarrows \mathrm{Fun}(\mathcal{C}, \mathcal{D}).$$

Indeed, it is easy to write the unit and counit for the adjunction on the functor categories from the one on the ordinary categories.

Next we claim that the composite

$$\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathrm{Fun}(I, \mathcal{D})) \cong \mathrm{Fun}(I, \mathrm{Fun}(\mathcal{C}, \mathcal{D}))$$

where the first functor comes from the constant functor $\Delta\colon \mathcal{D} \to \mathrm{Fun}(I, \mathcal{D})$ coincides with the constant functor

$$\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(I, \mathrm{Fun}(\mathcal{C}, \mathcal{D})).$$

Indeed, a functor $A\colon \mathcal{C} \to \mathcal{D}$ is sent under Δ to the functor

$$x \in \mathcal{C} \mapsto (i \in I \mapsto A(x)).$$

Viewed as a functor $I \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ this is

$$i \in I \mapsto (x \in \mathcal{C} \mapsto A(x)),$$

i.e. the constant functor.

But then the functor $\mathrm{Fun}(\mathcal{C}, \mathrm{Fun}(I, \mathcal{D})) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is right adjoint to the constant functor and is hence the limit functor.

The dual statement is proved similarly. $\square$

For instance, suppose $\mathcal{C}$ is a small category. Since Set has small limits and colimits, the presheaf category

$$\mathrm{PShv}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$$

also has small limits and colimits.

Recall that the Yoneda functor $Y\colon \mathcal{C} \to \mathrm{PShv}(\mathcal{C})$ is given by $x \mapsto \mathrm{Hom}_{\mathcal{C}}(-, x)$. Since limits in the functor category are computed pointwise, Theorem 4.3 implies that the Yoneda functor preserves limits. However, there is no reason to expect that it preserves colimits; in fact, the opposite is true: the presheaf category is a free cocompletion of $\mathcal{C}$. To explain this notion, let us first state the following straightforward lemma.

Given a presheaf $F\colon \mathcal{C}^{op} \to \mathrm{Set}$ we denote by $(* \Rightarrow F)^{op}$ the comma category whose objects are pairs (x, y) with $x \in \mathcal{C}$ and $y \in F(x)$ and morphisms being morphisms $x_1 \to x_2$ in $\mathcal{C}$ which induce maps of pointed sets $F(x_2) \to F(x_1)$. We have a functor

$$P\colon (* \Rightarrow F)^{op} \to \mathrm{PShv}(\mathcal{C})$$

which assigns $P(x, y) = Y(x)$ and the diagonal functor

$$\Delta\colon \mathrm{PShv}(\mathcal{C}) \to \mathrm{Fun}((* \Rightarrow F)^{op}, \mathrm{PShv}(\mathcal{C})).$$

Note that if $\mathcal{C}$ is small, so is the comma category $(* \Rightarrow F)^{op}$.

Lemma 4.8. *Let $\mathcal{C}$ be a small category. Then for any two presheaves F, G we have a natural isomorphism*

$$\mathrm{Hom}_{\mathrm{PShv}(\mathcal{C})}(F, G) \cong \mathrm{Hom}_{\mathrm{Fun}((* \Rightarrow F)^{op}, \mathrm{PShv}(\mathcal{C}))}(P, \Delta(G)).$$

Proof. Suppose that η belongs to $\mathrm{Hom}_{\mathrm{PShv}(\mathcal{C})}(F, G)$, so that η is a natural transformation from F to G. We define a corresponding natural transformation ζ in $\mathrm{Hom}_{\mathrm{Fun}((* \Rightarrow F)^{op}, \mathrm{PShv}(\mathcal{C}))}(P, \Delta(G))$ by

$$\zeta_{(x,y)}(g) = G(g)(\eta_x(y))$$

for any object (x, y) of $(* \Rightarrow F)^{op}$ and any g in $\mathrm{Hom}_{\mathcal{C}}(x', x)$. Conversely using the definition of a natural transformation we see that any ζ in $\mathrm{Hom}_{\mathrm{Fun}((* \Rightarrow F)^{op}, \mathrm{PShv}(\mathcal{C}))}(P, \Delta(G))$ must be of this form for a unique η in $\mathrm{Hom}_{\mathrm{PShv}(\mathcal{C})}(F, G)$. $\square$

Since colimits are left adjoint to the constant functor, we conclude the following.

Corollary 4.9. *Every presheaf is a colimit of representables. More precisely, for a presheaf F we have an isomorphism*

$$F \cong \mathrm{colim}_{(* \Rightarrow F)^{op}} P.$$

One says that the presheaf category is the free cocompletion of $\mathcal{C}$. It is *not* true that the formation of presheaves is a left adjoint to some forgetful functor due to size issues, but we have the following statement.

Corollary 4.10. *Let $\mathcal{C}$ be a small category and $\mathcal{D}$ a category which has small colimits. Then we have an equivalence of categories*

$$\mathrm{Fun}^{\mathrm{colim}}(\mathrm{PShv}(\mathcal{C}), \mathcal{D}) \cong \mathrm{Fun}(\mathcal{C}, \mathcal{D}),$$

where the category on the left is the full subcategory of functors which preserve small colimits.

Proof. Given a functor $\mathrm{PShv}(\mathcal{C}) \to \mathcal{D}$ we can restrict it to representable presheaves to get a functor $\mathcal{C} \to \mathcal{D}$. Conversely, given a functor $F\colon \mathcal{C} \to \mathcal{D}$ we define the colimit-preserving functor $\widehat{F}\colon \mathrm{PShv}(\mathcal{C}) \to \mathcal{D}$ as follows. For any presheaf $G \in \mathrm{PShv}(\mathcal{C})$ we can use Corollary 4.9 to get a presentation as a colimit:

$$G \cong \mathrm{colim}_{(*\Rightarrow G)^{op}} P.$$

We define

$$\widehat{F}(G) = \mathrm{colim}_{(*\Rightarrow G)^{op}} F \circ P.$$

Note that P lands in the subcategory of representable presheaves, so the expression $F \circ P$ makes sense.

We need to check that the functor $\widehat{F}$ commutes with colimits. On the one hand, for a diagram of presheaves G_i we have

$$\widehat{F}(\mathrm{colim}_i\, G_i) := \mathrm{colim}_{(*\Rightarrow \mathrm{colim}_i G_i)^{op}} F \circ P.$$

On the other hand,

$$\mathrm{colim}_i\, \widehat{F}(G_i) := \mathrm{colim}_i\, \mathrm{colim}_{(*\Rightarrow G_i)^{op}} F \circ P.$$

It is easy to see that both colimits are isomorphic, so $\widehat{F}$ preserves small colimits.

Suppose $G \cong Y(x)$ for some $x \in \mathcal{C}$. Then

$$\widehat{F}(Y(x)) = \mathrm{colim}_{(*\Rightarrow Y(x))^{op}} F \circ P \cong F(x)$$

as the category $(* \Rightarrow Y(x))^{op}$ has a final object (x, id_x). Therefore, the composite

$$\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}^{\mathrm{colim}}(\mathrm{PShv}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

is naturally isomorphic to the identity.

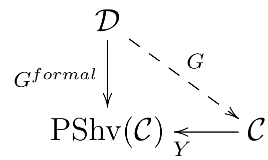
Suppose $\tilde{F}\colon \mathrm{PShv}(\mathcal{C}) \to \mathcal{D}$ is a functor preserving small colimits. Then it is uniquely determined on the subcategory of representable presheaves as any presheaf is a small colimit of representables. This shows that the composite

$$\mathrm{Fun}^{\mathrm{colim}}(\mathrm{PShv}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}^{\mathrm{colim}}(\mathrm{PShv}(\mathcal{C}), \mathcal{D})$$

is naturally isomorphic to the identity. □

4.3. **Adjoint functor theorem.** We have seen (Proposition 4.4) that if a functor $F\colon \mathcal{C}\to\mathcal{D}$ admits a right adjoint, it has to preserve colimits. However, even if $\mathcal{C}$ has small limits and F preserves colimits, the right adjoint might not exist due to set-theoretic issues. An adjoint functor theorem asserts that under some smallness assumptions if F preserves colimits it has a right adjoint (and dually for left adjoints). In this section we will present several versions of the adjoint functor theorem.

Let $F\colon \mathcal{C}\to\mathcal{D}$ be a functor. Recall that its right adjoint G can be defined as a lift of the formal right adjoint $G^{formal}\colon \mathcal{D}\to \mathrm{PShv}(\mathcal{C})$ along the Yoneda embedding



where the formal right adjoint is defined as $G^{formal}(y)(x) = \mathrm{Hom}_{\mathcal{D}}(Fx, y)$ for $y\in\mathcal{D}$ and $x\in\mathcal{C}$.

We could try to construct the right adjoint G universally if we had a functor $\mathrm{PShv}(\mathcal{C})\to\mathcal{C}$. The Yoneda embedding preserves limits but not colimits, so we might expect it to have a left adjoint.

Definition 4.11. A category $\mathcal{C}$ is *total* if the Yoneda embedding $Y\colon \mathcal{C}\to\mathrm{PShv}(\mathcal{C})$ has a left adjoint.

We denote by Y^L the left adjoint. If $P\in\mathrm{PShv}(\mathcal{C})$ is a presheaf, then

$$Y^L(P)\cong \operatorname*{colim}_{\substack{x\in\mathcal{C}\\ z\in P(x)}} x.$$

Conversely, the category $\mathcal{C}$ is total if this colimit exists.

Examples of total categories include the category of sets and the category of vector spaces (more generally, locally presentable categories which we will define shortly).

Theorem 4.12. *Suppose $F\colon \mathcal{C}\to\mathcal{D}$ is a functor between locally small categories with $\mathcal{C}$ total. Then F preserves colimits iff it has a right adjoint.*

Proof. We define the right adjoint by

$$G = Y^L G^{formal}.$$

Explicitly, for $y\in\mathcal{D}$ the object Gy is the colimit

$$Gy = \operatorname*{colim}_{\substack{x\in\mathcal{C}\\ z\in\mathrm{Hom}(Fx,y)}} x.$$

Unpacking the definition of the colimit we see that one has maps

$$\mathrm{Hom}(Fx,y)\to\mathrm{Hom}(x,Gy)$$

natural in x and y. This allows one to define the unit of the adjunction. To define the counit we just need to exhibit an element of $\mathrm{Hom}(FGy,y)$ which maps to $\mathrm{id}_{Gy}\in\mathrm{Hom}(Gy,Gy)$.

We have

$$\begin{aligned}\mathrm{Hom}(FGy, y) &\cong \mathrm{Hom}(F \operatorname{colim}_{x,z} x, y)\\ &\cong \mathrm{Hom}(\operatorname{colim}_{x,z} Fx, y)\\ &\cong \lim_{\substack{x\in\mathcal{C}\\ z\in\mathrm{Hom}(Fx,y)}} \mathrm{Hom}(Fx, y)\end{aligned}$$

There is a canonical element in the limit which assigns $z \in \mathrm{Hom}(Fx, y)$ to every $z \in \mathrm{Hom}(Fx, y)$. By tracing through the isomorphisms, it is easy to see that the composite

$$\lim_{x,z} \mathrm{Hom}(Fx, y) \to \lim_{x,z} \mathrm{Hom}(x, Gy) \cong \mathrm{Hom}(Gy, Gy)$$

sends this canonical element to id_{Gy}. □

It might be difficult to check that a given category is total, but many locally small categories with small colimits are locally presentable in the following sense.

Definition 4.13. A category is *essentially small* if it is equivalent to a small category.

Definition 4.14. A locally small category $\mathcal{C}$ is called *locally presentable* if there is an essentially small subcategory $\mathcal{C}^c \hookrightarrow \mathcal{C}$ of *compact objects* such that the functor $\mathcal{C} \to \mathrm{Fun}_{lex}(\mathcal{C}^{c,op}, \mathrm{Set})$ is an equivalence.

Here the functor category Fun_{lex} is the subcategory of functors which preserve finite limits, i.e. ones that send finite colimits in $\mathcal{C}^c$ to finite limits in Set.

Here are some examples of locally presentable categories:

- $\mathcal{C} = \mathrm{Set}$. Compact objects are finite sets.
- $\mathcal{C} = \mathrm{Vect}$. Compact objects are finite-dimensional vector spaces.
- $\mathcal{C} = \mathrm{Grp}$. Compact objects are finitely presented groups.

Here is an alternative characterization.

Proposition 4.15. *A category $\mathcal{C}$ is locally presentable iff it has a fully faithful embedding $i\colon \mathcal{C} \hookrightarrow \mathrm{PShv}(\mathcal{D})$ for $\mathcal{D}$ an essentially small category such that i commutes with filtered colimits and it has a left adjoint.*

If $\mathcal{C}$ is locally presentable, the embedding is given by $\mathrm{Fun}_{lex}(\mathcal{C}^{c,op}, \mathrm{Set}) \hookrightarrow \mathrm{PShv}(\mathcal{C}^c)$.

This proposition explains the name locally presentable: the category of presheaves $\mathrm{PShv}(\mathcal{C}^c)$ is obtained by freely adding colimits of objects of $\mathcal{C}^c$ and the left adjoint $\mathrm{PShv}(\mathcal{C}^c) \to \mathcal{C}$ imposes certain relations thus giving a presentation for objects of $\mathcal{C}$.

Theorem 4.16. *Suppose $\mathcal{C}$ is a locally presentable category and $\mathcal{D}$ a locally small category. Then a functor $F\colon \mathcal{C} \to \mathcal{D}$ has a right adjoint iff it preserves small colimits.*

Proof. Suppose F preserves colimits.

Let $\mathcal{C}^c \subset \mathcal{C}$ be the subcategory of compact objects. We can define a functor

$$G\colon \mathcal{D} \to \mathrm{PShv}(\mathcal{C}^c)$$

by $G(y)(x) = \mathrm{Hom}_{\mathcal{D}}(Fx, y)$ for $y \in \mathcal{D}$ and $x \in \mathcal{C}^c$. Since F preserves colimits, G lands in the subcategory $\mathrm{Fun}_{lex}(\mathcal{C}^{c,op}, \mathrm{Set}) \cong \mathcal{C}$.

By the Yoneda lemma we have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(x, Gy) &\cong G(y)(x)\\ &= \mathrm{Hom}_{\mathcal{D}}(Fx, y)\end{aligned}$$

for any $x \in \mathcal{C}^c$ and $y \in \mathcal{D}$.

Since any presheaf is a colimit of representables, the same formula holds for $x \in \mathcal{C}$ by writing any such element as a colimit of compact objects and observing that F preserves colimits. ☐

Let us finally state the most general adjoint functor theorem due to Freyd. Recall Proposition 2.8 which asserts that a right adjoint to $F\colon \mathcal{C} \to \mathcal{D}$ exists iff we can find a final object in the comma category $(F \Rightarrow x)$ for every $x \in \mathcal{D}$. A final object is a limit over the empty diagram. But we can also write it as a colimit over the identity functor which, however, is not a small colimit unless the category itself is small. In other words, we could find the final object if we are able to deal with some set-theoretic issues by reducing the colimit to a small colimit. This is the content of the so-called "solution set condition".

Lemma 4.17. *Let $\mathcal{C}$ be a locally small category with small colimits. Then $\mathcal{C}$ has a final object iff the following condition is satisfied:*

- *(Solution set condition) There is a small set I and a family of objects $\{c_i\}_{i\in I}$ of $\mathcal{C}$ such that for every $x \in \mathcal{C}$ there is an $i \in I$ so that $\mathrm{Hom}_{\mathcal{C}}(x, c_i)$ is non-empty.*

Proof.

- $\Rightarrow$.
  Suppose $\mathcal{C}$ has a final object. Then we take $I = *$ and c to be the final object. By the definition of final objects $\mathrm{Hom}_{\mathcal{C}}(x, c)$ consists of a single element for every $x \in \mathcal{C}$.
- $\Leftarrow$.
  Conversely, suppose the solution set condition is satisfied and let $w = \coprod_{i\in I} c_i$. Define c to be the coequalizer of all endomorphisms of w (since $\mathcal{C}$ is locally small, $\mathrm{End}(w)$ is a small set). Our goal is to prove that c is final.

  For any object $x \in \mathcal{C}$ there is a morphism $x \to w$ given by the composite $x \to c_i \to w$ for some $i \in I$. Post-composing with $p_c\colon w \to c$ we get a morphism $x \to c$.

  Suppose $f, g\colon x \to c$ are two morphisms and let $d = \mathrm{coeq}(f, g\colon x \to c)$ be their coequalizer. We denote by $p_d\colon c \to d$ the projection morphism. As before, we have a morphism $a\colon d \to w$. The composite

  $$w \xrightarrow{p_c} c \xrightarrow{p_d} d \xrightarrow{a} w$$

  defines an endomorphism of w. Since c is coequalizes any two endomorphisms of w, in particular id_w and $a \circ p_d \circ p_c$, we have

  $$p_c = p_c \circ a \circ p_d \circ p_c.$$

  Since p_c is an epimorphism (Proposition 3.24), we have

  $$\mathrm{id}_c = p_c \circ a \circ p_d.$$

  Similarly,

  $$p_d = p_d \circ p_c \circ a \circ p_d$$

which implies that

$$\mathrm{id} = p_d \circ p_c \circ a,$$

i.e. p_d is an isomorphism. Since d was a coequalizer of $x \rightrightarrows c$ we conclude that $f = g$, i.e. the set $\mathrm{Hom}(x, c)$ has a single element.

□

The solution set condition can be informally formulated as an assertion that even though the category $\mathcal{C}$ is large, every object can be detected from a small set of objects. With this lemma we can formulate the adjoint functor theorem.

Theorem 4.18. *Suppose $F\colon \mathcal{C} \to \mathcal{D}$ is a functor between locally small categories where $\mathcal{C}$ has small colimits. Then it has a right adjoint iff if preserves colimits and $(F \Rightarrow x)$ satisfies the solution set condition for all $x \in \mathcal{D}$.*

Proof.

- $\Rightarrow$.
  If F has a right adjoint, it has to preserve colimits by Proposition 4.4. The counit defines a final object by Proposition 2.8, so we see that $(F \Rightarrow x)$ satisfies the solution set condition.
- $\Leftarrow$.
  Suppose $(F \Rightarrow x)$ satisfies the solution set condition for all $x \in \mathcal{D}$. In view of Lemma 4.17 and Proposition 2.8 to conclude that F has a right adjoint we need to show that the comma category $(F \Rightarrow x)$ has small colimits.
  By Theorem 3.16 it is enough to show $(F \Rightarrow x)$ has small coproducts and coequalizers. By Proposition 4.6 it is enough to show that the projection functor $P\colon (F \Rightarrow x) \to \mathcal{C}$ creates respective colimits as $\mathcal{C}$ has small colimits.
  Suppose J is a small set and consider a collection $\{(y_j, f_j)\}_{j\in J}$ of objects of $(F \Rightarrow x)$ where $y_j \in \mathcal{C}$ and $f_j\colon F(y_j) \to x$. Consider a coproduct $\sqcup y_j$ in $\mathcal{C}$. Since F preserves coproducts, we have a unique morphism $F(\sqcup y_j) \cong \sqcup F(y_j) \to x$ making the diagram

$$\begin{array}{ccc} F(y_j) & & \\ \downarrow & \searrow & \\ F(\sqcup y_j) & \longrightarrow & x \end{array}$$

  commute. It is easy to show that this cocone is universal since $\sqcup y_j$ is universal. In the same way one shows that P creates coequalizers and so $(F \Rightarrow x)$ has small colimits.

□

Let us give an example of the application of a dual statement to Theorem 4.18. Consider the forgetful functor $F\colon \mathrm{Grp} \to \mathrm{Set}$; we will show that its left adjoint (the functor of the free group on a set) exists. Since F creates limits and Set has small limits, we just need to check the solution set condition for $(X \Rightarrow F)$ for all sets X. Consider a group G and a map of sets $X \to F(G)$. Each such map factors through the subgroup of G generated by the images of elements of X. Such a subgroup has cardinality bounded in terms of cardinality of X, so we can take as the generating set the set of representatives of each isomorphism class of such subgroups.

5. Monads

5.1. **Definitions.** The easiest category one encounters is the one-object category $*/M$ for M a monoid. One completely understands this category if one understands the monoid. One can freely complete this category with respect to colimits, i.e. consider $\mathrm{Fun}((*/M)^{op}, \mathrm{Set})$; this is the category of sets equipped with an action of M. Even though this is some large category we completely understand it since we understand what sets are and what an action of M is. In general a category involves a lot of data such as the set of objects, morphisms and composition maps. This might be difficult to understand especially if the category is constructed in an inexplicit way such as through some universal construction. Monads and in particular the Barr–Beck theorem gives us a way to reduce the complexity of categories by realizing them as categories of modules; once we understand the simpler category (such as the category of sets, vector spaces etc) and the algebra acting on it (called a monad in this case), we completely understand the original category.

Definition 5.1. A *monad* acting on a category $\mathcal{C}$ is an endofunctor $T\colon \mathcal{C} \to \mathcal{C}$ together with two natural transformations called *multiplication* $\mu\colon T \circ T \Rightarrow T$ and *unit* $\eta\colon \mathrm{id}_{\mathcal{C}} \Rightarrow T$ making the following diagrams commutative:

- (Associativity)

$$\begin{array}{ccc} T\circ T\circ T & \xRightarrow{\mathrm{id}\circ\mu} & T\circ T \\ {\scriptstyle \mu\circ\mathrm{id}}\Big\Downarrow & & \Big\Downarrow{\scriptstyle\mu} \\ T\circ T & \xRightarrow{\quad\mu\quad} & T \end{array}$$

- (Unit)

$$\begin{array}{ccc} T\circ \mathrm{id} & \xRightarrow{\mathrm{id}\circ\eta} & T\circ T \\ \Big\| & \swarrow\!\!\swarrow{\scriptstyle\mu} & \\ T & & \end{array}$$

and

$$\begin{array}{ccc} \mathrm{id}\circ T & \xRightarrow{\eta\circ\mathrm{id}} & T\circ T \\ \Big\| & \swarrow\!\!\swarrow{\scriptstyle\mu} & \\ T & & \end{array}$$

Note that in older texts a monad is called a triple.

Here is a basic example of a monad. Let A be a unital associative algebra. Then the functor $T\colon \mathrm{Vect} \to \mathrm{Vect}$ given by $V \mapsto A \otimes V$ has a structure of a monad. Indeed, the multiplication and the unit natural transformations come from the corresponding maps on A; the axioms are equivalent to those of a unital associative algebra.

Dually, one has comonads. That is, comonads on $\mathcal{C}$ are monads on $\mathcal{C}^{op}$.

Definition 5.2. A *comonad* acting on $\mathcal{C}$ is an endofunctor $T\colon \mathcal{C} \to \mathcal{C}$ together with two natural transformations called *comultiplication* $\Delta\colon T \Rightarrow T \circ T$ and *counit* $\epsilon\colon T \Rightarrow \mathrm{id}_{\mathcal{C}}$ satisfying the coassociativity and counit axioms dual to those of monads.

For instance, if A is a coalgebra, the functor $V \mapsto A \otimes V$ defines a comonad on vector spaces with the comultiplication and counit being those on A.

Another large source of examples of (co)monads comes from adjunctions.

Proposition 5.3. *Let $F\colon \mathcal{C} \rightleftarrows \mathcal{D} : G$ be an adjunction $F \dashv G$. Then FG has a structure of a comonad on $\mathcal{D}$ and GF has a structure of a monad on $\mathcal{C}$.*

Proof. Let $T = GF$ be an endofunctor of $\mathcal{C}$. The unit of T is given by the unit of the adjunction

$$\eta\colon \mathrm{id}_{\mathcal{C}} \Rightarrow GF.$$

The multiplication μ is induced by the counit $\epsilon\colon FG \Rightarrow \mathrm{id}_{\mathcal{D}}$:

$$\mu\colon GFGF \overset{\mathrm{id}_G \circ \epsilon \circ \mathrm{id}_F}{\Rightarrow} GF.$$

The axioms are easily checked using string diagrams. We will give an algebraic proof.

First, consider a diagram

$$\begin{array}{ccc} FGFG & \overset{\mathrm{id}_{FG} \circ \epsilon}{\Longrightarrow} & FG \\ {\scriptstyle \epsilon \circ \mathrm{id}_{FG}} \Downarrow & & \Downarrow {\scriptstyle \epsilon} \\ FG & \overset{\epsilon}{\Longrightarrow} & \mathrm{id}_{\mathcal{D}} \end{array}$$

Commutativity of a diagram of natural transformations is checked on objects, and for any $x \in \mathcal{D}$ the diagram

$$\begin{array}{ccc} FGFG(x) & \xrightarrow{FG(\epsilon_x)} & FG(x) \\ {\scriptstyle \epsilon_{FG(x)}} \downarrow & & \downarrow {\scriptstyle \epsilon_x} \\ FG(x) & \xrightarrow{\epsilon_x} & x \end{array}$$

is commutative due to naturality of ϵ.

We can compose the diagram with G on the left and F on the right to obtain a commutative diagram

$$\begin{array}{ccc} GFGFGF & \overset{\mathrm{id}_{GFG} \circ \epsilon \circ \mathrm{id}_F}{\Longrightarrow} & GFGF \\ {\scriptstyle \epsilon \circ \mathrm{id}_{FG}} \Downarrow & & \Downarrow {\scriptstyle \mathrm{id}_G \circ \epsilon \circ \mathrm{id}_F} \\ GFGF & \overset{\mathrm{id}_G \circ \epsilon \circ \mathrm{id}_F}{\Longrightarrow} & GF \end{array}$$

This is exactly the associativity diagram for T.

One of the unit diagrams is

$$\begin{array}{ccc} GF & \overset{\eta}{\Longrightarrow} & GFGF \\ \| & \swarrow\!\!\swarrow {\scriptstyle \mathrm{id}_G \circ \epsilon \circ \mathrm{id}_F} & \\ GF & & \end{array}$$

Commutativity of this diagram is equivalent to one of the adjunction axioms. The commutativity of the other unit axiom is equivalent to the other adjunction axiom. $\square$

5.2. **Algebras.** Given a monoid a natural notion to understand is that of a module over the monoid. For monads this notion is called an algebra over a monad.

Definition 5.4. Let $T\colon \mathcal{C} \to \mathcal{C}$ be a monad. An *algebra* over T is an object $x \in \mathcal{C}$ together with a morphism $a\colon Tx \to x$ in $\mathcal{C}$ which satisfies the following two axioms:

- (Associativity) The diagram

$$\begin{array}{ccc} T^2x & \xrightarrow{T(a)} & Tx \\ {\scriptstyle \mu_x}\downarrow & & \downarrow{\scriptstyle a} \\ Tx & \xrightarrow{a} & x \end{array}$$

  commutes.
- (Unit) The composite

$$x \stackrel{\eta_x}{\to} Tx \stackrel{a}{\to} x$$

  is the identity.

We denote by $\mathrm{Alg}_T(\mathcal{C})$ the category of T-algebras in $\mathcal{C}$. The objects are T-algebras (x, a) and morphisms $(x, a_x) \to (y, a_y)$ are given by morphisms $f\colon x \to y$ making the diagram

$$\begin{array}{ccc} Tx & \xrightarrow{a_x} & x \\ {\scriptstyle T(f)}\downarrow & & \downarrow{\scriptstyle f} \\ Ty & \xrightarrow{a_y} & y \end{array}$$

commute.

Dually one defines *coalgebras* over comonads and we denote by $\mathrm{CoAlg}_T(\mathcal{C})$ the category of T-coalgebras in $\mathcal{C}$.

Recall that an algebra A defines a monad $V \mapsto A \otimes V$ on vector spaces. The category of algebras over this monad is equivalent to the category of A-modules.

Here is an example of a different kind. We have a monad on Set given by sending a set X to the set of finite words on X. The unit is given by the inclusion of one-letter words and the multiplication is given by concatenation: a word on a set of words on a set X can be written as a word on X. Algebras over this monad are the same as unital monoids: the action map is given by the composition of the word within the monoid. The fact that this definition is equivalent to the usual compact definition of monoids in terms of an associative composition rule is an example of a coherence theorem.

Given a monad T on $\mathcal{C}$ a reasonable question is whether it can be presented as a monad associated to an adjunction. It turns out to always be the case.

Proposition 5.5. *Let T be a monad acting on $\mathcal{C}$. Then the forgetful functor $F\colon \mathrm{Alg}_T(\mathcal{C}) \to \mathcal{C}$ has a left adjoint and the associated monad on $\mathcal{C}$ is naturally isomorphic to T. Dually, for a comonad T the forgetful functor $\mathrm{CoAlg}_T(\mathcal{C}) \to \mathcal{C}$ has a right adjoint and the associated comonad on $\mathcal{C}$ is naturally isomorphic to T.*

Proof. Let T be a monad $\mathcal{C} \to \mathcal{C}$. We define a functor $L\colon \mathcal{C} \to \mathrm{Alg}_T(\mathcal{C})$ by $x \mapsto Tx$. The T-algebra structure on Tx is given by $T^2x \stackrel{\mu_x}{\to} Tx$. It is immediate that the axioms for T-algebras are satisfied.

Let (A, a_A) be a T-algebra and $x \in \mathcal{C}$. Then a morphism of T-algebras $Tx \to A$ is the same as a morphism $f \colon Tx \to A$ in $\mathcal{C}$ such that the bottom square in the diagram

$$\begin{array}{ccc} Tx & & \\ {\scriptstyle T(\eta_x)}\downarrow & \searrow\!\!= & \\ T^2x & \xrightarrow{\mu_x} & Tx \\ {\scriptstyle T(f)}\downarrow & & \downarrow{\scriptstyle f} \\ TA & \xrightarrow{a_A} & A \end{array}$$

is commutative. The commutativity of the top triangle follows from the unit axiom of the monad.

But then the morphism $f \colon Tx \to A$ is given by the composite

$$Tx \overset{T(\eta_x)}{\to} T^2x \overset{T(f)}{\to} TA \overset{a_A}{\to} A.$$

In other words, f is uniquely determined by the composite

$$x \overset{\eta_x}{\to} Tx \overset{f}{\to} A.$$

Hence

$$\mathrm{Hom}_{\mathrm{Alg}_T(\mathcal{C})}(Lx, A) \cong \mathrm{Hom}_{\mathcal{C}}(x, FA).$$

By definition $FL(x) = Tx$. □

Let $F \colon \mathcal{C} \rightleftarrows \mathcal{D} : G$ be an adjunction with $F \dashv G$. We have the associated monad $T = GF$ on $\mathcal{C}$ and a comonad $S = FG$ on $\mathcal{D}$. Moreover, we get natural functors

$$G_{enh} \colon \mathcal{D} \to \mathrm{Alg}_T(\mathcal{C})$$

and

$$F_{enh} \colon \mathcal{C} \to \mathrm{CoAlg}_S(\mathcal{D})$$

such that the diagrams of categories

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{G} & \mathcal{C} \\ {\scriptstyle G_{enh}}\downarrow & \nearrow{\scriptstyle forget} & \\ \mathrm{Alg}_{GF}(\mathcal{C}) & & \end{array}$$

and

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ {\scriptstyle F_{enh}}\downarrow & \nearrow{\scriptstyle forget} & \\ \mathrm{CoAlg}_{FG}(\mathcal{D}) & & \end{array}$$

naturally commute.

For instance $G_{enh} \colon \mathcal{D} \to \mathrm{Alg}_T(\mathcal{C})$ is given by $y \mapsto Gy$. Its T-algebra structure is given by

$$GFGy \overset{G(\epsilon_y)}{\to} Gy.$$

Definition 5.6. A functor $G \colon \mathcal{D} \to \mathcal{C}$ is *monadic* if it has a left adjoint and for the corresponding monad T the functor $\mathcal{D} \to \mathrm{Alg}_T(\mathcal{C})$ is an equivalence.

Definition 5.7. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is *comonadic* if it has a right adjoint and for the corresponding comonad S the functor $\mathcal{C} \to \mathrm{CoAlg}_S(\mathcal{D})$ is an equivalence.

Our goal in the next subsection will be to characterize (co)monadic functors. Let us first make few preliminary observations.

Proposition 5.8. *Let T be a monad on $\mathcal{C}$. Then the forgetful functor $\mathrm{Alg}_T(\mathcal{C}) \to \mathcal{C}$ creates limits. If T preserves colimits of shape I, then the forgetful functor reflects those colimits.*

Proof. Consider a diagram $J\colon I \to \mathrm{Alg}_T(\mathcal{C})$ and denote the forgetful functor by $F\colon \mathrm{Alg}_T(\mathcal{C}) \to \mathcal{C}$. To show that F creates limits, we have to show that $\lim_I FJ$ has a unique T-algebra structure compatible with the forgetful maps.

That is, we have to construct and show uniqueness of the map $T(\lim_I FJ) \to \lim_I FJ$ making the diagrams

$$\begin{array}{ccc} T(\lim_I FJ) & \longrightarrow & \lim_I FJ \\ \downarrow & & \downarrow \\ TFJ(i) & \longrightarrow & FJ(i), \end{array}$$

where at the bottom we use the T-algebra maps on $J(i)$. By the universal property of the limit every such map is determined uniquely. It is then straightforward to check that thus defined T-algebra satisfies a universal property.

Now suppose T preserves colimits of shape I and consider a diagram $J\colon I \to \mathrm{Alg}_T(\mathcal{C})$. We are going to define a T-algebra structure on $\mathrm{colim}_I FJ$. The action map is given by

$$T(\mathrm{colim}_I FJ) \cong \mathrm{colim}_I TFJ \to \mathrm{colim}_I FJ.$$

The universal property of this T-algebra is immediate from the universality of the colimit. □

5.3. **Barr–Beck.** In this subsection we will prove a recognition theorem for monadic functors, i.e. we will prove necessary and sufficient conditions for a functor to be monadic. The corresponding statement is known as the Barr–Beck theorem (also called Beck's monadicity theorem).

Before we state the Barr–Beck theorem, we need two preliminary notions: that of a conservative functor and a split coequalizer.

Definition 5.9. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is *conservative* if it reflects isomorphisms, i.e. if for some morphism $f\colon x \to y$ the morphism $F(f)$ is an isomorphism, so is f.

For instance, many forgetful functors are conservative: Vect → Set, Grp → Set, Alg → Vect etc.

Here is a basic lemma about conservative functors.

Lemma 5.10. *Suppose $F\colon \mathcal{C} \to \mathcal{D}$ is a conservative functor which admits either a fully faithful left adjoint or a fully faithful right adjoint. Then F is an equivalence.*

Proof. We assume that F admits a fully faithful left adjoint. The other statement is proved by duality.

Let F^L be the left adjoint. Since F^L is fully faithful, the unit

$$\eta\colon \mathrm{id}_{\mathcal{D}} \Rightarrow FF^L$$

is a natural isomorphism. To show F is an equivalence we have to show that the counit $\epsilon\colon F^L F \Rightarrow \mathrm{id}_{\mathcal{C}}$ is also an isomorphism. By one of the adjunction axioms we know that the composite

$$F(x) \stackrel{\eta_{F(x)}}{\to} FF^LF(x) \stackrel{F(\epsilon_x)}{\to} F(x)$$

is $\mathrm{id}_{F(x)}$ for every $x \in \mathcal{C}$. This implies that $F(\epsilon_x)$ is an isomorphism and by conservativity ϵ_x is an isomorphism. □

Next we discuss split pairs of morphisms. Let $f, g\colon x \to y$ be a pair of morphisms in a category $\mathcal{C}$. A *fork* is a cocone

$$x \underset{f}{\overset{g}{\rightrightarrows}} y \stackrel{e}{\to} z.$$

We say that a fork is *split* if there exist morphisms $s\colon z \to y$ and $t\colon y \to x$ such that

$$es = \mathrm{id}_z, \quad ft = \mathrm{id}_y, \quad gt = se.$$

Proposition 5.11. *Every split fork is a coequalizer.*

Proof. Suppose $w \in \mathcal{C}$ and we have a morphism $e'\colon y \to w$ such that

$$e'f = e'g.$$

We want to prove there is a unique morphism $h\colon z \to w$ such that $he = e'$. Uniqueness is clear: we have

$$h = hes = e's.$$

Now suppose $h = e's$. Then $he = e'se = e'gt = e'ft = e'$. □

Definition 5.12. A pair of morphisms $f, g\colon x \to y$ is called a *split pair* if their coequalizer exists and is split.

Let $F\colon \mathcal{C} \to \mathcal{D}$ be some functor. Then we say that a pair f, g of morphisms in $\mathcal{C}$ is F*-split* if the pair $F(f), F(g)$ is split.

Here are two important examples of split pairs:

(1) Let $T\colon \mathcal{C} \to \mathcal{C}$ be a monad with $\mu\colon T^2 \Rightarrow T$ the multiplication and $x \in \mathcal{C}$ be a T-algebra with the action map $a\colon Tx \to x$. Then the pair

$$T^2x \underset{T(a)}{\overset{\mu_x}{\rightrightarrows}} Tx$$

is split.

Indeed, we have a fork

$$T^2x \underset{T(a)}{\overset{\mu_x}{\rightrightarrows}} Tx \stackrel{a}{\to} x$$

due to the associativity axiom of the action a.

We claim that $s = \eta_x\colon x \to Tx$ and $t = \eta_{Tx}\colon Tx \to T^2x$ give a splitting. Indeed,

$$a\eta_x = \mathrm{id}_x$$

by the unit axiom of a T-algebra. The equation

$$\mu_x \eta_{Tx} = \mathrm{id}_{Tx}$$

is the unit axiom of the monad and

$$T(a)\eta_{Tx} = \eta_x a$$

holds by naturality of η.

(2) Now let $F \dashv G$ be an adjunction where $F \colon \mathcal{C} \rightleftarrows \mathcal{D} : G$. We claim that for every $y \in \mathcal{D}$ the fork

$$FGFG(y) \underset{FG(\epsilon_y)}{\overset{\epsilon_{FG(y)}}{\rightrightarrows}} FG(y) \overset{\epsilon_y}{\to} y$$

is G-split. Indeed, after applying G we get a fork

$$GFGFG(y) \underset{GFG(\epsilon_y)}{\overset{G(\epsilon_{FG(y)})}{\rightrightarrows}} GFG(y) \overset{G(\epsilon_y)}{\to} G(y)$$

which is a split fork for the GF-algebra $G(y)$ by the previous example.

Theorem 5.13 (Barr–Beck)**.** *A functor $G \colon \mathcal{D} \to \mathcal{C}$ is monadic iff the following conditions are satisfied:*

- *G admits a left adjoint,*
- *G is conservative,*
- *Every G-split pair of morphisms admits a coequalizer in $\mathcal{D}$ and it is preserved by G.*

Proof. Consider the diagram

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{G} & \mathcal{C} \\ {\scriptstyle G_{enh}}\big\downarrow & \nearrow{\scriptstyle \text{forget}} & \\ \mathrm{Alg}_T(\mathcal{C}) & & \end{array}$$

where $T = GF$.

- $\Rightarrow$.

  Suppose G is a monadic functor, i.e. G_{enh} is an equivalence. Then G is conservative iff the forgetful functor $\mathrm{Alg}_T(\mathcal{C}) \to \mathcal{C}$ is conservative. Suppose $x, y \in \mathrm{Alg}_T(\mathcal{C})$ are two T-algebras together with a morphism of T-algebras $f \colon x \to y$ which is an isomorphism in $\mathcal{C}$. That is, the diagram

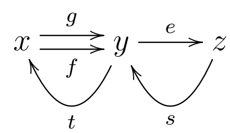
$$\begin{array}{ccc} Tx & \xrightarrow{T(f)} & Ty \\ \big\downarrow & & \big\downarrow \\ x & \xrightarrow[f]{} & y \end{array}$$

  is commutative. But then the diagram

$$\begin{array}{ccc} Tx & \xleftarrow[T(f)^{-1}]{} & Ty \\ \big\downarrow & & \big\downarrow \\ x & \xrightarrow[f^{-1}]{} & y \end{array}$$

  is commutative and hence f^{-1} is a morphism of T-algebras, i.e. f is an isomorphism in $\mathrm{Alg}_T(\mathcal{C})$ and so G is conservative.

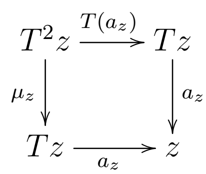
Now suppose $x, y \in \mathrm{Alg}_T(\mathcal{C})$ are two T-algebras with a pair of morphisms of T-algebras $f, g\colon x \to y$. Moreover, assume we have a split fork



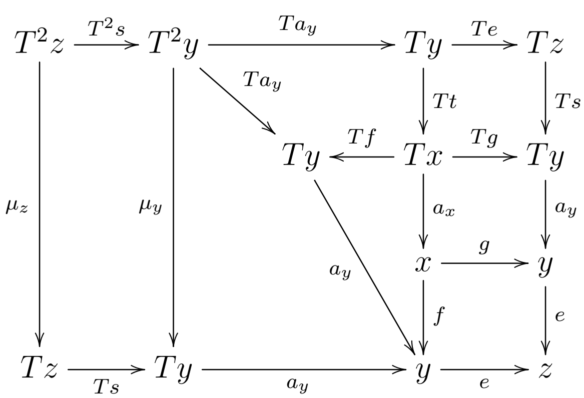
where s, t, e are merely morphisms in $\mathcal{C}$ while f, g are morphisms of T-algebras. Let us now prove that z is also a T-algebra and e is a morphism of T-algebras. Define the T-action a_z on z as the composite

$$Tz \xrightarrow{T(s)} Ty \xrightarrow{a_y} y \xrightarrow{e} z.$$

Associativity of z is the commutativity of the diagram

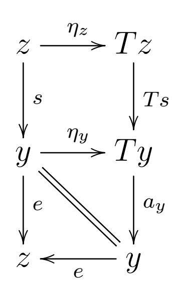


We can expand it to a diagram



The commutativity of the subdiagrams follows from the associativity and fork axioms. For instance, the top trapezoid commutes since $ft = \mathrm{id}_y$. Therefore, the action map a_z is associative.

The unit axiom follows from the commutative diagram



since $es = \mathrm{id}_z$. We also get a commutative diagram

$$\begin{array}{ccc} Ty & \xrightarrow{T(e)} & Tz \\ \downarrow & & \downarrow \\ y & \xrightarrow{e} & z \end{array}$$

and so e is a morphism of T-algebras.

Finally, we have to show that z is a coequalizer. Consider a T-algebra w together with a morphism $e'\colon y \to w$ such that $e'f = e'g$. Since z is a coequalizer in $\mathcal{C}$, there is a unique morphism $h\colon z \to w$ in $\mathcal{C}$ given by $h = e's$ and we just need to show that h is a morphism of T-algebras. This is equivalent to the commutativity of the diagram

$$\begin{array}{ccccccc} Tz & \xrightarrow{T(s)} & Ty & \xrightarrow{a_y} & y & \xrightarrow{e} & z \\ & & \downarrow{\scriptstyle T(e')} & & \downarrow{\scriptstyle e'} & & \downarrow{\scriptstyle s} \\ & & Tw & \xrightarrow{a_w} & w & \xleftarrow{e'} & y \end{array}$$

Commutativity of the first square follows from the fact that e' is a morphism of T-algebras and the commutativity of the second square follows from the chain of equalities

$$e'se = e'gt = e'ft = e'.$$

- $\Leftarrow$. [2]

  Conversely, suppose the conditions of the theorem are satisfied. To prove that G is monadic, i.e. that G_{enh} is an equivalence, by Lemma 5.10 it is enough to show that G_{enh} is conservative and admits a fully faithful left adjoint. Clearly, conservativity of G implies that of G_{enh}: if $f\colon x \to y$ is a morphism in $\mathcal{D}$ such that $G_{enh}(f)\colon G_{enh}(x) \to G_{enh}(y)$ is an isomorphism of T-algebras, then $G(f)$ is an isomorphism in $\mathcal{C}$ and hence f is an isomorphism.

  Recall that the forgetful functor forget: $\mathrm{Alg}_T(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint free which sends an object $x \in \mathcal{C}$ to the free T-algebra Tx. The putative left adjoint G^L_{enh} is thus uniquely determined on free T-algebras:

$$G^L_{enh}(Tx) \cong F(x)$$

  for any $x \in \mathcal{C}$. The idea of the proof is to resolve every T-algebra by free T-algebras and apply this prescription to the resolution to construct the left adjoint for all T-algebras.

  Let $\mathcal{A} \subset \mathrm{Alg}_T(\mathcal{C})$ be the full subcategory of objects satisfying the following two conditions:

  (1) For every $x \in \mathcal{A}$ the functor

$$\mathcal{D} \to \mathrm{Set}$$

  given by $y \mapsto \mathrm{Hom}_{\mathrm{Alg}_T(\mathcal{C})}(x, G_{enh}y)$ is corepresentable.

[2] This proof is adapted from Lemma 4.7.4.13 in J. Lurie, *Higher algebra*.

(2) The unit morphism $\eta_x\colon x \to G_{enh}G^L_{enh}(x)$ for every $x \in \mathcal{A}$ induces an isomorphism

$$\mathrm{forget}(x) \to \mathrm{forget}(G_{enh}(G^L_{enh}(x))) \cong G(G^L_{enh}(x)).$$

Then we can define a functor $G^L_{enh}\colon \mathcal{A} \to \mathcal{D}$ which is a partial left adjoint to G_{enh}:

$$\mathrm{Hom}_{\mathcal{D}}(G^L_{enh}(x), y) \cong \mathrm{Hom}_{\mathrm{Alg}_T(\mathcal{C})}(x, G_{enh}(y))$$

naturally in $x \in \mathcal{A}$ and $y \in \mathcal{D}$. We will show that $\mathcal{A}$ is equivalent to $\mathrm{Alg}_T(\mathcal{C})$ and so the left adjoint to G_{enh} exists.

First, the essential image of free: $\mathcal{C} \to \mathrm{Alg}_T(\mathcal{C})$ is in $\mathcal{A}$:

$$\mathrm{Hom}_{\mathrm{Alg}_T(\mathcal{C})}(\mathrm{free}(x), G_{enh}(y)) \cong \mathrm{Hom}_{\mathcal{C}}(x, Gy) \cong \mathrm{Hom}_{\mathcal{D}}(Fx, y)$$

for $x \in \mathcal{C}$ and $y \in \mathcal{D}$ and the unit

$$\mathrm{free}(x) \to G_{enh}G^L_{enh}\mathrm{free}(x)$$

induces an isomorphism

$$\mathrm{forget}(\mathrm{free}(x)) \to \mathrm{forget}(G_{enh}(G^L_{enh}(\mathrm{free}(x)))) \cong GF(x) = T(x).$$

By the example preceding this theorem, every T-algebra x fits into a coequalizer

$$T^2x \underset{T(a_x)}{\overset{\mu_x}{\rightrightarrows}} Tx \xrightarrow{a_x} x$$

of free T-algebras. We denote by x_i the corresponding diagram $T^2x \rightrightarrows Tx$. Moreover, we've shown that $\mathrm{forget}(x_i) \cong GG^L_{enh}(x_i)$ defines a split coequalizer. By the assumptions of the theorem, we see that $G_{enh}(x_i)$ is G-split and so has a coequalizer in $\mathcal{D}$.

Then for every $x \in \mathrm{Alg}_T(\mathcal{C})$ and $y \in \mathcal{D}$ we have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Alg}_T(\mathcal{C})}(x, G_{enh}(y)) &\cong \mathrm{Hom}_{\mathrm{Alg}_T(\mathcal{C})}(\mathrm{colim}_i\, x_i, G_{enh}(y)) \\ &\cong \lim_i \mathrm{Hom}_{\mathrm{Alg}_T(\mathcal{C})}(x_i, G_{enh}(y)) \\ &\cong \lim_i \mathrm{Hom}_{\mathcal{D}}(G^L_{enh}(x_i), y) \\ &\cong \mathrm{Hom}_{\mathcal{D}}(\mathrm{colim}_i\, G^L_{enh}(x_i), y).\end{aligned}$$

For $x \in \mathrm{Alg}_T(\mathcal{C})$ the morphism

$$\mathrm{forget}(x) \to GG^L_{enh}(x)$$

is given by the composite

$$\mathrm{colim}_i\, x_i \cong \mathrm{colim}_i\, GG^L_{enh}(x_i) \to G(\mathrm{colim}_i\, G^L_{enh}(x_i)) \cong GG^L_{enh}(x)$$

which is an isomorphism since $G^L_{enh}(x_i)$ is G-split and G preserves such coequalizers.

In this way we've defined the left adjoint $G^L_{enh}\colon \mathrm{Alg}_T(\mathcal{C}) \to \mathcal{D}$ to G_{enh} such that the unit

$$\eta_x\colon x \to G_{enh}G^L_{enh}(x)$$

is an isomorphism after applying the functor forget. But the latter functor is conservative, so the unit is an isomorphism. Therefore, G^L_{enh} is fully faithful and since G_{enh} is conservative we see that G_{enh} is an equivalence.

□

One application of the Barr–Beck theorem is to Tannakian formalism (this is due to Deligne). Namely, it is a formalism allowing one to recognize which categories are categories of representations of an algebraic group.

5.4. **Descent.** In this section we will apply the Barr–Beck theorem to the problem of descent first in Galois theory (extensions of fields) and then more generally to commutative algebra (extensions of rings). We will be quite sketchy here just to outline the general idea. We will apply the comonadic version of the Barr–Beck theorem which is dual to 5.13, so let us state it here.

Theorem 5.14. *A functor $F\colon \mathcal{C} \to \mathcal{D}$ is comonadic iff the following conditions are satisfied:*

- *F admits a right adjoint,*
- *F is conservative,*
- *Every F-cosplit pair of morphisms admits an equalizer in $\mathcal{C}$ and it is preserved by F.*

In the statement we have used the term "F-cosplit pair" to denote the notion dual to that of a split pair. This will be irrelevant in this subsection as all categories will have equalizers and the functors F will preserve them.

Let us begin with the baby case of descent which is descent for functions. Given a set X we denote by $\mathcal{O}(X)$ the commutative k-algebra of k-valued functions on X for some fixed field k which will be implicit. Given a map of sets $X \to Y$ we can precompose functions with this map to obtain a pullback morphism $\mathcal{O}(Y) \to \mathcal{O}(X)$. In this way we get a contravariant functor

$$\mathcal{O}\colon \mathrm{Set}^{op} \to \mathrm{CAlg}$$

from sets to commutative algebras. Since $\mathcal{O}$ is the algebra of functions, it is immediate that $\mathcal{O}$ preserves limits, i.e. it sends colimits in Set to limits in CAlg.

Now, if $X \to Y$ is a surjective map, the canonical map

$$\mathrm{coeq}(X \times_Y X \rightrightarrows X) \to Y$$

is an isomorphism simply by explicitly computing the coequalizer (here the two maps $X \times_Y X \rightrightarrows X$ are the two natural projections). Therefore, we get that the pullback $\mathcal{O}(Y) \to \mathcal{O}(X)$ induces an isomorphism of commutative algebras

$$\mathcal{O}(Y) \xrightarrow{\sim} \mathrm{eq}(\mathcal{O}(X) \rightrightarrows \mathcal{O}(X \times_Y X)).$$

In other words, functions on Y can be written in terms of functions on X satisfying a descent condition that their two pullbacks to $\mathcal{O}(X \times_Y X)$ are equal. Descent theory generalizes this to morphisms of some geometric objects where we replace the vector spaces $\mathcal{O}(-)$ by certain categories associated to the geometric objects. In that case descent conditions become descent data which have to satisfy further coherences.

5.4.1. *Galois descent.* We begin with the easiest case of descent which is that of vector spaces over fields. Let $k \subset L$ be an extension of fields. Then on the categories of vector spaces we get an adjunction

$$\mathrm{Ind}\colon \mathrm{Vect}_k \rightleftarrows \mathrm{Vect}_L\colon \mathrm{Res},$$

where the left adjoint Ind is given by sending a k-vector space V to the L-vector space $V \otimes_k L$ and the right adjoint Res is given by considering an L-vector space as a k-vector space via the embedding $k \subset L$. This adjunction is an instance of the tensor-Hom adjunction of modules over rings that we have discussed before.

Let us try to apply the Barr–Beck theorem to this adjunction. Both categories Vect_k and Vect_L have all limits and colimits. Clearly, Res is conservative. So, to prove that Ind is conservative we just need to show that the composite Res ∘ Ind is conservative. Suppose $f\colon V \to W$ is a linear map of k-vector spaces which induces an isomorphism

$$V \otimes_k L \to W \otimes_k L$$

of *k-vector spaces*. Since $L \cong \oplus_i k$ as a k-vector space we see that

$$\oplus_i V \stackrel{\oplus_i f}{\to} \oplus_i W$$

is an isomorphism. But this implies that f itself is an isomorphism.

An equalizer $V \underset{f}{\overset{g}{\rightrightarrows}} W$ in Vect_k is the same as a kernel of $f - g$, so we just need to show that Ind preserves kernels, i.e. the natural map

$$\ker(f) \otimes_k L \to \ker(f \otimes_k L)$$

is an isomorphism for any linear map $f\colon V \to W$. This is proved again by writing $L \cong \oplus_i k$ as k-vector spaces.

Therefore, Ind satisfies the conditions of the Barr–Beck theorem 5.14 and so Ind is comonadic. The comonad $S = \mathrm{Ind} \circ \mathrm{Res}$ on Vect_L is given by

$$V \mapsto S(V) = V \otimes_k L \cong V \otimes_L L \otimes_k L$$

and the Barr–Beck theorem asserts that Vect_k is equivalent to S-coalgebras in Vect_L. But this comonad is simply given by tensoring with the L-coalgebra $L \otimes_k L$ and hence Vect_k is equivalent to the category of $L \otimes_k L$-comodules in Vect_L. Note that the coproduct

$$L \otimes_k L \to L \otimes_k L \otimes_L L \otimes_k L \cong L \otimes_k L \otimes_k L$$

is given by $l_1 \otimes l_2 \mapsto l_1 \otimes 1 \otimes l_2$ and the counit

$$L \otimes_k L \to L$$

is $l_1 \otimes l_2 \mapsto l_1 l_2$.

Definition 5.15. A *k-form* of an L-vector space V is a k-vector space V_k together with an isomorphism $V_k \otimes_k L \cong V$.

Therefore, the Barr–Beck theorem implies the following proposition.

Proposition 5.16. *Let $k \subset L$ be a field extension and V an L-vector space. Then the category of k-forms of V is equivalent to the category of coactions of $L \otimes_k L$ on V.*

Let us now make further assumptions on the field extension to recover a more familiar statement from Galois theory. First we make an assumption that the extension $k \subset L$ is finite, i.e. L is a finite-dimensional vector space over k. Then the coalgebra $L \otimes_k L$ is finite-dimensional. Its dual L-algebra is $\mathrm{Hom}_L(L \otimes_k L, L) \cong \mathrm{Hom}_k(L, L)$. The previous statement can then be reformulated as this: k-forms of V are equivalent to actions of $\mathrm{Hom}_k(L, L)$ on V. To understand this algebra, let us recall the following definition.

Definition 5.17. A finite field extension $k \subset L$ is *Galois* if $L^{\mathrm{Aut}(L/k)} = k$.

Proposition 5.18. *A finite field extension $k \subset L$ is Galois iff the natural morphism $L[\mathrm{Aut}(L/k)] \to \mathrm{Hom}_k(L, L)$ is an isomorphism.*

We recover the following theorem from Galois theory.

Theorem 5.19. *Let $k \subset L$ be a finite Galois extension. Then k-forms of an L-vector space V are equivalent to actions of the Galois group $\mathrm{Aut}(L/k)$ on V.*

5.4.2. *Faithfully flat descent.* Let us generalize the descent picture from fields to commutative rings. Suppose R, S are commutative rings with a morphism $R \to S$. Recall that the functor $S \otimes_R -\colon R-\mathrm{mod} \to S-\mathrm{mod}$ commutes with colimits since it has a right adjoint. It also commutes with finite products since those coincide with finite coproducts in the categories of modules.

Definition 5.20. A morphism $f\colon R \to S$ is *faithfully flat* if the functor

$$\mathrm{Ind} = S \otimes_R -\colon R-\mathrm{mod} \to S-\mathrm{mod}$$

is faithful and preserves equalizers.

Assume that we are given a faithfully flat morphism $R \to S$. We are going to apply the Barr–Beck theorem to the tensor product functor Ind to show that it's comonadic.

The tensor-Hom adjunction asserts that the right adjoint to $\mathrm{Ind}\colon R-\mathrm{mod} \to S-\mathrm{mod}$ is the functor $\mathrm{Hom}_S(S, -)\colon S-\mathrm{mod} \to R-\mathrm{mod}$ which is simply the restriction functor Res from S-modules to R-modules.

To show that Ind is conservative, consider a morphism of R-modules $g\colon M \to N$ and suppose $S \otimes_R M \to S \otimes_R N$ is an isomorphism. The coequalizer of g and the zero morphism is simply the cokernel of g; similarly, the equalizer of g and 0 is the kernel of g. By assumption the tensor product functor preserves kernels and cokernels, so we just need to show that if $\mathrm{Ind}(P) \cong 0$ for some R-module P, then $P \cong 0$. Indeed, for any R-module Q we have an injective map $\mathrm{Hom}_R(Q, P) \to \mathrm{Hom}_S(\mathrm{Ind}(Q), \mathrm{Ind}(P)) \cong \{0\}$. Therefore, there is a unique morphism into P from any R-module, so P has to be a final object and hence $P \cong 0$. Therefore, Ind is conservative.

The comonad $\mathrm{Ind} \circ \mathrm{Res}$ is given by $M \mapsto S \otimes_R M \cong S \otimes_R S \otimes_S M$. We get the following statement from the Barr–Beck theorem.

Proposition 5.21. *The category of R-modules is equivalent to the category of S-modules M equipped with a coassociative coaction morphism $M \to S \otimes_R M$.*

This statement is usually formulated as follows: modules over rings satisfy faithfully flat descent.

It turns out that while faithfully flat morphisms of rings are sufficient for descent, they are not necessary. The necessary and sufficient condition is given by the notion of a pure monomorphism of rings.

Definition 5.22. A morphism of R-modules $M_1 \to M_2$ is a *pure monomorphism* if for every R-module N the induced morphism $N \otimes_R M_1 \to N \otimes_R M_2$ is injective.

Let us show that if $R \to S$ is a faithfully flat morphism of rings, it is a pure monomorphism of R-modules. That is, we have to show that

$$N \to N \otimes_R S$$

is injective for every R-module N. By faithful flatness of $R \to S$ it is enough to show that $N \otimes_R S \to N \otimes_R S \otimes_R S$ is injective, but this is split: there is the multiplication map $N \otimes_R S \otimes_R S \to N \otimes_R S$ given by $n \otimes s_1 \otimes s_2 \mapsto n \otimes s_1 s_2$ which splits the inclusion.

It turns out that being a pure monomorphism is the optimal condition for descent (see [JT, Corollary 5.4]).

Theorem 5.23. *The tensor product functor* $S \otimes_R -\colon R-\mathrm{mod} \to S-\mathrm{mod}$ *is comonadic iff the morphism of* R*-modules* $R \to S$ *is a pure monomorphism.*

6. Categorical algebraic geometry

Functors of points. In this section we sketch how category theory is applied in modern algebraic geometry. Many details will be omitted; see for example [EH] for a much more thorough introduction.

Algebraic geometry is concerned with viewing solutions of polynomial equations as geometric objects. For instance, we can consider the set $C(\mathbb{Z})$ of integers (x, y) satisfying the equation

$$y^2 = x^3 - 1.$$

Notice, however, that it makes sense to consider solutions where x and y belong to any ring. The assignment to any ring the set of solutions of this equation defines a functor

$$C\colon \mathrm{Rng} \to \mathrm{Set}.$$

The set of solutions of the equation

$$y^2 = x^3 - \pi$$

cannot be made sense of in any ring since we need to have π in the ring as well. However, this equation defines a functor

$$\mathrm{Alg}_{\mathbb{C}} \to \mathrm{Set}$$

from the category of $\mathbb{C}$-algebras to sets.

So, let's fix a *base ring* k. In the above examples $k = \mathbb{Z}$ or $\mathbb{C}$. The above examples show that geometric objects can be considered as functors.

Definition 6.1. A *space* is a functor $\mathrm{Alg}_k \to \mathrm{Set}$.

Let $\mathrm{Sp} \equiv \mathrm{Fun}(\mathrm{Alg}_k, \mathrm{Set})$ be the category of spaces. The approach to algebraic geometry via such functors is known as the *functor of points* approach. The value of a functor on a ring R is called the set of R-points of the space.

We have the Yoneda embedding

$$\mathrm{Alg}_k^{op} \to \mathrm{Sp}$$

and we define the essential image of the Yoneda embedding to be the category of *affine schemes* Aff. Since the Yoneda embedding is fully faithful,

$$\mathrm{Aff} \cong \mathrm{Alg}_k^{op}.$$

We may tautologically rewrite spaces as contravariant functors $\mathrm{Aff}^{op} \to \mathrm{Set}$, i.e. spaces are just presheaves on the category of affine schemes:

$$\mathrm{Sp} \cong \mathrm{PShv}(\mathrm{Aff}).$$

The Yoneda embedding in terms of affine schemes is called the spectrum functor

$$\mathrm{Spec}\colon \mathrm{Aff} \to \mathrm{Sp}.$$

The examples we have given above are both affine schemes. To show that, we need to prove that the functor

$$\mathrm{Rng} \to \mathrm{Set}$$

sending a ring to the set of solutions of $y^2 - x^3 + 1 = 0$ is corepresented by a ring. Indeed, consider the quotient ring $R = \mathbb{Z}[x, y]/(y^2 - x^3 + 1)$. A morphism of rings $f\colon R \to S$ is uniquely determined by specifying the elements $f(x), f(y) \in S$ satisfying the equation

$$f(y)^2 = f(x)^3 - 1.$$

Thus, $C \cong \mathrm{Spec}\, R$, i.e. it is an affine scheme.

Functions. What are possible functions on $\mathrm{Spec}\, R$? We are allowed to measure x and y, which are not independent but have to satisfy the equation $y^2 = x^3 - 1$. In other words, if we denote by $\mathcal{O}(X)$ the ring of functions on a space X, then we expect

$$\mathcal{O}(\mathrm{Spec}\, R) \cong R.$$

We will take it as a definition of the ring of functions on an affine scheme. How do we determine functions on an arbitrary space X? For any morphism $f\colon X \to Y$ of spaces we expect to be able to restrict functions $f^*\colon \mathcal{O}(Y) \to \mathcal{O}(X)$. But any presheaf is determined by morphisms from representable presheaves, so we might define $\mathcal{O}(X)$ as follows. A function $g \in \mathcal{O}(X)$ is an assignment to a pair (R, f), where $R \in \mathrm{Alg}_k$ and $f\colon \mathrm{Spec}\, R \to X$, the function $f^*g \in R$. These have to be compatible in the sense that if we have (R_1, f) and a morphism $h\colon R_1 \to R_2$, then $h(f^*g) = (f \circ h)^*g$. More concisely, we define the functor $\mathcal{O}\colon \mathrm{Sp} \to \mathrm{Rng}^{op}$ to be

$$\mathcal{O}(X) = \lim_{\mathrm{Spec}\, R \to X} R.$$

It is immediate that the functor $\mathcal{O}$ sends colimits in Sp to limits in Rng, so we could instead take it as a defining property and use the fact that every presheaf is a colimit of representables.

An important example of an affine scheme is an affine space $\mathbb{A}^n$. It is defined to be the functor which sends a ring R to the set R^n whose elements are given by sequences of n elements of R. Alternatively,

$$\mathbb{A}^n \cong \mathrm{Spec}\, k[x_1, ..., x_n].$$

Another important example of an affine scheme is $\mathbb{G}_m$. It is defined by $\mathbb{G}_m(R) = R^\times$, the set of invertible elements (i.e. units) of R. It is also an affine scheme and we have

$$\mathbb{G}_m \cong \mathrm{Spec}\, k[x, x^{-1}].$$

So far all examples we've given were affine schemes. Let us now give a natural example of a non-affine scheme. Before we define it, we will need a notion of an invertible module.

Definition 6.2. A module M over a commutative ring R is *invertible* if there is an R-module N such that $M \otimes_R N \cong R$.

For instance, the module $M = R$ is invertible with the inverse given by itself.

We define the projective space $\mathbb{P}^n$ to be the functor which assigns to R the set of invertible R-submodules of $R^{\oplus(n+1)}$. If $R = k$ is a field, all invertible modules are isomorphic to k. In this case $\mathbb{P}^n(k)$ is the set of lines (i.e. one-dimensional subspaces) in k^{n+1} which gives the classical notion of a projective space associated to a given vector space k^{n+1}.

Proposition 6.3. *$\mathbb{P}^n$ is not an affine scheme. In fact, $\mathcal{O}(\mathbb{P}^n) \cong k$.*

Note that since Spec sends colimits in Rng to limits in Sp (Yoneda embedding preserves limits), we have

$$\operatorname{Spec} R_1 \times_{\operatorname{Spec} S} \operatorname{Spec} R_2 \cong \operatorname{Spec}(R_1 \otimes_S R_2).$$

Tangent spaces. All geometric properties of spaces can be completely encoded in the corresponding functor of points. Let us give a single example. Let $X \in \mathrm{Sp}$ be a space and let $x \in X(k)$ be a k-point. Let us denote by $\pi\colon k[\epsilon]/\epsilon^2 \to k$ the natural projection.

Definition 6.4. The *tangent space* $T_x X$ of X at x is the fibre of $X(\pi)\colon X(k[\epsilon]/\epsilon^2) \to X(k)$ at $x \in X(k)$.

Exercise. Show that $T_x X$ is a k-vector space.

For instance, let $L \in \mathbb{P}^n(k)$ be a line in k^{n+1}. Then $T_L \mathbb{P}^n \cong k^{n+1}/L$.

Quasi-coherent sheaves. Since $\mathcal{O}(\mathbb{P}^n) \cong k$, we see that the set of global functions on a space is not a very good approximation to the space if the space is not an affine scheme. Instead, one might consider certain categorical functions by replacing elements of R by R-modules. Here is the definition.

Definition 6.5. Let X be a space. A *quasi-coherent sheaf* $\mathcal{F}$ on X is the following collection of data:

- For each ring R and $f\colon \operatorname{Spec} R \to X$ an R-module $f^*\mathcal{F}$
- For every morphism $g\colon \operatorname{Spec} R_1 \to \operatorname{Spec} R_2$ of affine schemes and $f\colon \operatorname{Spec} R_2 \to X$ an isomorphism $f^*\mathcal{F} \otimes_{R_2} R_1 \cong (f \circ g)^*\mathcal{F}$.

The isomorphisms have to satisfy the cocycle condition: for any pair of morphisms $g_1\colon \operatorname{Spec} R_1 \to \operatorname{Spec} R_2$ and $g_2\colon \operatorname{Spec} R_2 \to \operatorname{Spec} R_3$ and a morphism $f\colon \operatorname{Spec} R_3 \to X$ the two isomorphisms

$$(f \circ g_2 \circ g_1)^*\mathcal{F} \cong f^*\mathcal{F} \otimes_{R_3} R_1$$

and

$$(f \circ g_2 \circ g_1)^*\mathcal{F} \cong (f \circ g_2)^*\mathcal{F} \otimes_{R_2} R_1 \cong f^*\mathcal{F} \otimes_{R_3} R_2 \otimes_{R_2} R_1 \cong f^*\mathcal{F} \otimes_{R_3} R_1$$

are equal.

Even though $\mathcal{O}(\mathbb{P}^n) \cong k$, the category of quasi-coherent sheaves $\mathrm{QCoh}(\mathbb{P}^n)$ is non-trivial: it is a certain quotient of the category of graded modules over $k[x_1, ..., x_{n+1}]$, where $k[x_1, ..., x_{n+1}]$ is considered as a graded ring with each x_i in degree 1.

As for functions, one can write

$$\mathrm{QCoh}(X) = \widetilde{\lim}_{\operatorname{Spec} R \to X} R - \mathrm{mod},$$

where we take the limit in the 2-category of categories whose definition is beyond the scope of this course.

Exercise. Define $\mathrm{QCoh}(X)' = \lim_{\mathrm{Spec}\, R \to X} R - \mathrm{mod}$ where the limit is taken in the category of categories and compare the definition of objects of $\mathrm{QCoh}(X)'$ to the definition of quasi-coherent sheaves.

The faithfully flat descent property allows one to compute the category of quasi-coherent sheaves on a space by giving a faithfully flat cover by an affine scheme.

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