Analytic Topology: Problem sheet 4

1. Suppose X is a T_4 space and that $h : X \to \beta X$ is the "canonical embedding" of X into its Stone-Čech compactification. If A and B are disjoint closed subsets of X, prove that $\overline{h(A)}^{\beta X} \cap \overline{h(B)}^{\beta X} = \varnothing$.

2. Let E denote the set of even natural numbers. Show that $\overline{h(E)}^{\beta \mathbb{N}}$ is homeomorphic to βE, and hence that $\beta\mathbb{N}$ may be represented as the disjoint union of two homeomorphs of itself. (Here $\mathbb{N}$ has its usual discrete topology.)

3. If $\mathscr{N}_p$ is the neighbourhood filter in $\beta\mathbb{N}$ of a point $p \in \beta\mathbb{N} \setminus h(\mathbb{N})$, and $\mathscr{U} = \{N \cap \mathbb{N} : N \in \mathscr{N}_p\}$, show that $\mathscr{U}$ is a free ultrafilter on $\mathbb{N}$, and deduce that $\mathscr{N}_p$ does not have a countable basis.

4. Show that every compact metric space is a continuous image of $\beta\mathbb{N}$.

5. Is $\beta\mathbb{N}$ metrisable? (If so, prove it; if not, show why not.)

6. Let E be that subset of $\mathbb{R}^2$ defined by

$$E = \left[\bigcup_{n=1}^{\infty} \left([0,1] \times \left\{\frac{1}{n}\right\}\right)\right] \cup \{(0,0)\} \cup \{(1,0)\}.$$

Identify all the components and all the quasi-components of E. (E has the topology induced by the usual topology on $\mathbb{R}^2$.)

7. Let F be that subset of $\mathbb{R}^2$ defined by

$$F = \left\{(x,y) : y = \sin\frac{\pi}{x},\ x \in (0,1]\right\} \cup A, \text{ where } A = \{(0,y) : -1 \leq y \leq 1\},$$

with the topology induced by the usual topology on $\mathbb{R}^2$. Prove that F is a connected, but not locally connected, topological space.

8. Suppose that A and B are closed subsets of a compact Hausdorff space X such that no component of X meets both A and B. Show that there exists a clopen subset C of X such that $A \subseteq C \subseteq X \setminus B$.

9. Suppose that D is a non-empty compact proper subset of a connected Hausdorff space Y. Show that every component of D meets the boundary of D.

10. For $f, g \in \prod_{n\in\mathbb{N}}[0,1]$, define $D(f,g) = \sum_{n=0}^{\infty} 2^{-n}|f(n) - g(n)|$. Show that the Tychonoff topology on the product $\prod_{n\in\mathbb{N}}[0,1]$, and the topology generated by the metric D, are the same.

11. Prove that a paracompact regular space is normal.

12. Prove that a closed subspace of a paracompact space is paracompact.

13. Prove that the following three conditions on a regular space X are equivalent:

(i) Every open covering of X has a locally finite open refinement (that is, X is paracompact);

(ii) Every open covering of X has a locally finite refinement (the elements of the refinement not being necessarily open or closed),

(iii) Every open covering of X has a locally finite closed refinement (the elements of the refinement being closed sets).

[For (iii)⇒(i), it may help to do the following: Let $\mathscr{V}$ be an open cover witnessing local finiteness of a refinement. Now find a locally finite closed refinement of $\mathscr{V}$, and try to use it to build a locally finite open refinement of the original cover. Or, look it up in Willard!]

14. Let $\mathbb{B}$ be a Boolean algebra. Prove that the following hold, for all $a, b, c \in \mathbb{B}$:

(i) $a \leq b$ if and only if $a \vee b = b$,

(ii) $b = \neg a$ if and only if $b \wedge a = \mathbb{0}$ and $b \vee a = \mathbb{1}$,

(iii) $\neg(a \wedge b) = (\neg a) \vee (\neg b)$.

15. Let $\mathbb{A}$, $\mathbb{B}$ be Boolean algebras, and let $\phi : \mathbb{A} \to \mathbb{B}$ be a homomorphism. Prove that $\mathscr{S}\phi$ is one-to-one if and only if ϕ is onto, and is onto if and only if ϕ is one-to-one.

16. Let $\mathbb{B}$ be a Boolean algebra. Prove that $\eta_{\mathbb{B}} : \mathbb{B} \to \mathscr{B}\mathscr{S}\mathbb{B}$ is an isomorphism.

17. (i) What is the Stone dual of the one-point topological space?

(ii) Let $*$ be the one-point topological space. Let X be a compact Hausdorff zero-dimensional space. One can think of a point of X as being the range of a function from $*$ to X. Let $f : * \to X$, and let p be the unique point in the range of f. (f is automatically continuous. Why?) Describe $\mathscr{S}f$ completely in terms of p.

(iii) A *product* of two Boolean algebras $\mathbb{A}$ and $\mathbb{B}$ is the cartesian product with pointwise operations, that is, $\mathbb{1}_{\mathbb{A}\times\mathbb{B}} = (\mathbb{1}_{\mathbb{A}}, \mathbb{1}_{\mathbb{B}})$, $\mathbb{0}_{\mathbb{A}\times\mathbb{B}} = (\mathbb{0}_{\mathbb{A}}, \mathbb{1}_{\mathbb{B}})$, $\neg(a, b) = (\neg a, \neg b)$, $(a_1, b_1) \wedge (a_2, b_2) = (a_1 \wedge a_2, b_1 \wedge b_2)$, $(a_1, b_1) \vee (a_2, b_2) = (a_1 \vee a_2, b_1 \vee b_2)$, and $(a_1, b_1) \leq (a_2, b_2)$ if and only if $a_1 \leq b_1$ and $a_2 \leq b_2$. Suppose that Z is the disjoint union of two compact zero-dimensional Hausdorff spaces X and Y. Show that $\mathscr{B}Z$ is isomorphic to $\mathscr{B}X \times \mathscr{B}Y$.

Optional questions

Any extra material in this part of the question sheet is off the syllabus. These questions, some of which are hard, are intended for further study, or entertainment.

18. One statement of the *Baire Category Theorem* is that any intersection of countably many dense open subsets of $\mathbb{R}$ is non-empty.

(i) Deduce from the above statement that any intersection of countably many dense open subsets of $\mathbb{R}$ is dense.

(ii) We say that a subset of $\mathbb{R}$ is of *first category* if it is contained in the complement of an intersection of countably many dense sets. Intuitively, one can regard first category sets as being small, and the Baire Category Theorem says that $\mathbb{R}$ is not of first category. Prove that any countable union of first category sets is of first category and so is not small.

(iii) Prove the Baire Category Theorem.

[*Hint: Remember that any intersection of a strictly decreasing sequence of closed bounded intervals is non-empty.*]

(iv) (For those who know some measure theory.) Find a subset Q of $\mathbb{R}$ such that $\mathbb{R}\setminus Q$ is of first category, and Q is null.

19. (i) Let A be a subset of $\mathbb{R}$ having the property that for all points x of $\mathbb{R}$ except at most one, there exists an open neighbourhood U of x such that $A\cap U$ is countable. Prove that A is countable.

(ii) Deduce that if A is an uncountable subset of $\mathbb{R}$, there exist (at least) two distinct points x and y such that for every open neighbourhood U of either x or y, $U \cap A$ is uncountable.

(iii) Prove that every uncountable closed subset of $\mathbb{R}$ contains a homeomorphic copy of the Cantor set.

(iv) (Harder) Prove that any intersection of countably many dense open subsets of $\mathbb{R}$ contains a homeomorphic copy of the Cantor set.

20. (i) Let X be a compact zero-dimensional metric space. Note that X is second countable, by Urysohn's Metrisation Theorem. Prove that every clopen subset of X is a finite union of basic open sets. Prove that $\mathscr{B}X$ is countable.

(ii) Let $\mathbb{P}$ be $\wp\mathbb{N}$, considered as a Boolean algebra. Define $h : \mathbb{N} \to \mathscr{S}\mathbb{P}$ so that $h(n) = \{A \subseteq \mathbb{N} : n \in A\}$. Then h is continuous. (Why?) Let $f : \mathbb{N} \to [0,1]$ be any function. Define a continuous function $g : \mathscr{S}\mathbb{P} \to [0,1]$ so that for all $n \in \mathbb{N}$, $g(h(n)) = f(n)$, and deduce that $\mathscr{S}\mathbb{P}$ is homeomorpic to $\beta\mathbb{N}$.

[*Hint: Suppose that $p \in \mathscr{S}\mathbb{P}$. What is p? What value might you choose for $g(p)$?*]

(iii) (Much easier.) Now deduce that $\beta\mathbb{N}$ is not metrisable.

21. (Quite hard.) The *Sorgenfrey Line* $\mathbb{S}$ is the real line with the topology generated by sets of the form $(a,b]$, for real numbers a and b. The *Michael Line* $\mathbb{M}$ is the real line equipped with basic open sets of the form $\{r\}$ for irrational r and $(r - \frac{1}{n}, r + \frac{1}{n})$ for r rational and n a natural number.

(i) Prove that the Michael Line is T_4.

(ii) In the product $(\mathbb{R}\setminus\mathbb{Q})\times\mathbb{M}$, let C be the set of points (x,y) such that y is rational, and let D be the set of points (x,x), for x irrational. Prove that C and D are disjoint and closed, but that if U and V are any open sets such that $C \subseteq U$ and $D \subseteq V$, then $U \cap V \neq \varnothing$. Deduce that this product is not normal.

[*Intuitively, this is because, letting $V_{(x,x)}$ be a neighbourhood of the point (x,x) in D such that $V_{(x,x)}$ is contained in V, then too many of the sets $V_{(x,x)}$ are too big and crowd too close together so as to force U and V to intersect. To make this argument formal, you will need the Baire Category Theorem, from further up the sheet.*]

(iii) Prove that the Sorgenfrey Line is T_4.

(iv) Prove that in $\mathbb{S}\times\mathbb{S}$, any subset of the antidiagonal $\{(x,-x) : x \in \mathbb{R}\}$ is closed.

(v) In $\mathbb{S}\times\mathbb{S}$, let C be the set of points of the antidiagonal with rational coordinates, and let D be the set of points of the antidiagonal with irrational coordinates. Prove that if U and V are open sets such that $C \subseteq U$ and $D \subseteq V$, then $U \cap V \neq \varnothing$, and deduce that $\mathbb{S} \times \mathbb{S}$ is not normal.

22. (For those who know some set theory.) Assume ZFC.

(i) Prove that every uncountable closed subset of $\mathbb{R}$ has cardinality $2^{\aleph_0}$.

(ii) Prove that there are $2^{\aleph_0}$ closed subsets of $\mathbb{R}$. [*Hint: how many open subsets are there?*]

(iii) (Hard, and requiring some form of Choice.) A *Bernstein Set* is a subset B of $\mathbb{R}$ such that both B and its complement meet every uncountable closed set. Prove that a Bernstein set exists.