SOLUTIONS TO PROBLEM SHEET 4

Let X be a separable reflexive Banach space and $M \subset X$ be non-empty, closed and convex.

QUESTION 1. **Monotone Operators** Let $A \colon M \to X^*$ be a monotone operator.

(1) Using monotonicity first, and then Minty's Lemma, show that A satisfies condition (H3).

(2) Show that if A is strictly monotone, i.e. so that $\langle A(u) - A(v), u - v \rangle > 0$ for all $u \neq v$, then there exists at most one solution $u \in M$ of the variational inequality $\langle A(u), u - v \rangle \leq 0$ for all $v \in M$.

(3) Show that if A is strongly monotone, i.e. so that there exists $c > 0$ such that

$$\langle A(u) - A(v), u - v \rangle > c \|u - v\|^2,$$

and A maps bounded sets to bounded sets, then the variational inequality has a unique solution.

(4) Using the above show that the equation

$$-\Delta u = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega$$

has a unique solution for every $f \in L^2(\Omega)$, $\Omega \subset \mathbb{R}^n$ a bounded open subset with smooth boundary.

(1) By definition, an operator $A: M \to X^*$ is monotone if
- (monotonicity) $\langle A(u) - A(v), u - v \rangle \geq 0$
- (hemicontinuous) $\forall u, v \in M, w \in X$
  $t \mapsto \langle A(tu + (1-t)v), w \rangle$ is continuous on $[0,1]$.

WTS: A satisfies condition (H3), i.e.

(i) If $u_n \overset{w}{\rightharpoonup} u$ in X and $A(u_n) \overset{w}{\rightharpoonup} \xi$ in X^* then

$$\langle \xi, u \rangle \leq \underline{\lim} \langle A(u_n), u_n \rangle$$

(ii) equality in (i) $\Rightarrow$ $\langle A(u) - \xi, u - v \rangle \leq 0 \quad \forall v \in M$.

Proof Let $(u_n)_n \in M$ st $u_n \overset{w}{\rightharpoonup} u$ and $A(u_n) \to \xi$.

Note that, as X is reflexive $A(u_n) \overset{w}{\rightharpoonup} \xi \Leftrightarrow A(u_n) \overset{w^*}{\rightharpoonup} \xi$.

Then $\langle A(u_n)-\xi, v\rangle \to 0 \quad \forall v \in X$.

In particular

$$\langle A(u_n)-\xi, u\rangle \to 0$$

and also

$$\langle A(u), u_n - u\rangle \to 0 \text{ as } u_n \rightharpoonup u.$$

Thus, we have

$$\langle A(u_n), u_n - u\rangle = \underbrace{\langle A(u_n)-A(u), u_n-u\rangle}_{\substack{\geq 0 \\ \text{by monotonicity}}} + \underbrace{\langle A(u), u_n - u\rangle}_{\to 0}$$

giving

$$\underline{\lim} \langle A(u_n), u_n\rangle \geq \overline{\lim} \langle A(u_n), u\rangle = \langle \xi, u\rangle$$

which is (i) ✓

Now, let us assume that $\underline{\lim} \langle A(u_n), u_n\rangle = \langle \xi, u\rangle$

By Minty, (ii) $\Leftrightarrow$ $\langle A(v)-\xi, u-v\rangle \leq 0 \quad \forall v \in M$ $(\tilde{ii})$

so we proceed to prove $(\tilde{ii})$.

First, note that monotonicity implies

$$\langle A(v) - A(u_n), u_n - v\rangle \leq 0$$

Then

$$0 \geq \overline{\lim}\Big(\underbrace{\langle A(v), u_n\rangle}_{①} - \underbrace{\langle A(u_n), u_n\rangle}_{②} - \underbrace{\langle A(v), v\rangle}_{\text{fixed}} + \underbrace{\langle A(u_n), v\rangle}_{③}\Big) = \circledast$$

We note that:

- ① $\to \langle A(v), u\rangle$ by weak convergence of u_n

• ③ $\to \langle \xi, v \rangle$ by weak convergence of $A(u_n)$

• $\overline{\lim}\ -②\ = -\underline{\lim}\ ②\ = -\langle \xi, u \rangle$ by assumption

From where we get that

$$\circledast = \langle A(v), u \rangle - \langle \xi, u \rangle - \langle A(v), v \rangle + \langle \xi, v \rangle$$
$$= \langle A(v) - \xi, u - v \rangle$$

which is $(\widetilde{ii})$, and therefore, by Minty, we get (ii) ✓

□

(2) We proceed by contradiction. Assume u_1, u_2 are solutions of

$$\langle A(u_i), u_i - v \rangle \leq 0 \quad \forall v \in M, \ i = 1, 2.$$

Then

$$\langle A(u_1) - A(u_2), u_1 - u_2 \rangle = \langle A(u_1), u_1 - u_2 \rangle + \langle A(u_2), u_2 - u_1 \rangle \leq 0$$

and the fact that A is strictly monotone $\Rightarrow u_1 = u_2$.

(3) In order to prove that $\langle A(u), u - v \rangle \leq 0 \ \forall v \in M$ has a unique solution, we need to verify the conditions from Theorem 37 from Lecture Notes, ie (H1), (H2) and (H3).

(H1): A maps bounded sets to bounded sets $\to$ Holds by assumption

(H2): A is coercive w.r.t. some $u_0 \in M \to$ To check

(H3) $\to$ satisfied since A is strongly monotone ($\Rightarrow$ monotone $\Rightarrow$ (H3) by part (4)).

We therefore only need to verify (H2)

Proof of (H2): Fix any $u_0 \in M$. Then

$$\langle A(u), u-u_0\rangle = \underbrace{\langle A(u)-A(u_0), u-u_0\rangle}_{\geq C\|u-u_0\|^2} + \langle A(u_0), u-u_0\rangle$$

(by strong monotonicity)

On the other hand, as $A(u_0) \in X^*$, $\exists\, C_2 < \infty$ st

$$\langle A(u_0), u-u_0\rangle \leq C_2 \|u-u_0\|$$

$$\Rightarrow \langle A(u), u-u_0\rangle \geq C\|u-u_0\|^2 - C_2\|u-u_0\|$$

$$\frac{\langle A(u), u-u_0\rangle}{\|u-u_0\|} \geq C\|u-u_0\| - C_2 \longrightarrow \infty$$

when $\|u\| \to \infty$

Question 2. **Monotonicity, Convexity**

Let X be a Banach space and $F\colon X \to \mathbb{R}$ Gâteaux differentiable in every point $u \in X$ with Gâteaux derivative $F'(u)$. Show that

$$F \text{ is convex} \quad \Leftrightarrow \quad F'\colon X \to X^* \text{ is monotone.}$$

Remark:

- A map $G\colon X \to X^*$ is monotone if $\langle G(u) - G(v), u - v\rangle \geq 0$ for all $u, v \in X$ (i.e. hemicontinuity, as in the definition of a monotone operator, is not required).
- A function $F\colon X \to \mathbb{R}$ is convex on X, if $F(tu + (1-t)v) \leq tF(u) + (1-t)F(v)$ for all $t \in [0,1]$ and $u, v \in X$.
- Recall that a differentiable function $g : I \subset \mathbb{R} \to \mathbb{R}$ is convex on I if g' is monotonically increasing on I. Consider $g(t) := F(tu + (1-t)v)$.

Recall: F Gâteaux differentiable $\Leftrightarrow \forall x \in X \; \exists F'(x) \in X^*$ with $F'(x)(v) = \partial_v F(u) \quad \forall v \in X$.

(1) F convex $\Rightarrow F' : X \to X^*$ monotone

- F convex $\Rightarrow \forall u, v \in X$
  $t \mapsto g_{u,v}(t) := F(tu + (1-t)v)$ is convex.

- F Gâteaux-diff $\Rightarrow$ directional derivative exists, so $g_{u,v}$ is differentiable with

$$g_{u,v}'(t) = \langle F'(tu + (1-t)v), (u-v)\rangle$$

- $g_{u,v}$ convex $\Rightarrow g_{u,v}'$ non-decreasing

Then

$$0 \leq g_{u,v}'(1) - g_{u,v}'(0) = \langle F'(u), u - v\rangle - \langle F'(v), u - v\rangle$$

so F' monotone.

(2) $F' : X \to X^*$ monotone $\Rightarrow$ F convex

Let $u, v \in X$, and $g_{u,v}$ as above.

F' monotone $\Rightarrow g_{u,v}'(s) - g_{u,v}'(t) \geq 0$ for $s \geq t$.
Since

$$g_{u,v}'(s) - g_{u,v}'(t) = \langle F'(su + (1-s)v), u - v \rangle - \langle F'(tu + (1-t)v), u - v \rangle$$

Set $u_s := su + (1-s)v$ and note that $tu + (1-t)v = u_s + (t-s)(u-v)$
Then

$$g_{u,v}'(s) - g_{u,v}'(t) = \langle F'(u_s), u-v \rangle - \langle F'(u_s + (t-s)(u-v)), u-v \rangle$$
$$= \frac{1}{(s-t)} \underbrace{\langle F'(u_s) - F'(u_s + (t-s)(u-v)), (t-s)(u-v) \rangle}_{\geq 0 \text{ as } F' \text{ monotone}}$$

So g' non-decreasing. Thus g is convex and hence

$$F(tu + (1-t)v) \leq tF(u) + (1-t)F(v) \quad \forall t \in [0,1], \forall u, v \in X.$$

And consequently, F is convex.

Question 3. **Strongly monotone operator** Let $\Omega = (-1,1)$ and $X = H^2(\Omega) \cap H_0^1(\Omega)$ endowed with the H^2-norm.

(a) Let $A \colon X \to X^*$ be defined via

$$\langle A(u), v\rangle := \int_\Omega u''v''dx.$$

Show that A is a strongly monotone operator, i.e. hemicontinuous and so that there exists some $c_0 > 0$ with

$$\langle A(u) - A(v), u - v\rangle \geq c_0\|u - v\|^2 \quad \text{for all } u, v \in M.$$

Hint: Use Poincaré's inequality, as well as Poincaré's inequality for functions with mean value zero.

(b) Let now $F_\mu(u) := A(u) + \mu B(u)$ where $B(u)(v) := u(0)\cdot v(0) + \int_\Omega x \cdot v(x)dx$.

Show that $F_\mu : X \to X^*$ is well defined for any $\mu \in \mathbb{R}$ and that there exists a number $\mu_0 > 0$ so that for each μ with $|\mu| \leq \mu_0$ there exists a unique solution of the equation

$$F_\mu(u) = 0.$$

(c) Let now $\mu \geq 0$. Determine a functional $I_\mu : X \to \mathbb{R}$ so that the following holds: $u \in X$ is a solution of $F_\mu(u) = 0$ if and only if u is a minimiser of I_μ on X

(a) ▷ $A : X \to X^*$ is linear and bounded,

$$\|A(u)\|_{X^*} = \sup_{\|v\|_X = 1} |A(u)| \leq \|u''\|_{L^2(\Omega)} \leq \|u\|_X$$

$\Rightarrow A$ is continuous $\Rightarrow A$ also hemicontinuous

▷ WTS : $\langle A(u) - A(v), u - v\rangle \geq c_0 \|u-v\|^2 \quad \forall\, u,v \in M$

- By "H_0^1" version of Poincaré-ineq $\|u\|^2_{L^2(\Omega)} \leq C_1 \|u'\|^2_{L^2(\Omega)}$

- By "mean value zero" version of Poincaré-ineq

$$\|u'\|^2_{L^2(\Omega)} \leq C_2 \|u''\|^2_{L^2(\Omega)}$$

since $\displaystyle\int_\Omega u'\,dx = u(1) - u(-1) = 0 \quad \forall\, u \in H_0^1(\Omega).$

Combining these, we get $\forall u \in X$

$$\|u\|^2_{H^2(\Omega)} \leq (1+C_1)\|u'\|^2_{L^2(\Omega)} + \|u''\|^2_{L^2(\Omega)} \leq \underbrace{(C_2(1+C_1)+1)}_{=: 1/C_0} \underbrace{\|u''\|^2_{L^2(\Omega)}}_{= \langle A(u), u\rangle}$$

From this and the fact that A is linear, we also get

$$\langle A(u) - A(v), u-v\rangle = \langle A(u-v), u-v\rangle \geq C_0 \|u-v\|^2_{H^2(\Omega)}$$

proving that A is strongly monotone.

(1b) ▷ WTS: F_μ is well-defined

- Let $B_0(u)(v) = u(0)v(0)$.

Then $B_0 : X \to X^*$ is well-defined, linear and bounded $\Rightarrow$ continuous.

Moreover, as $H^2(\Omega) \hookrightarrow C^0(\Omega)$, we have

$$|u(0)| \leq C_3 \|u\|_{H^2(\Omega)}$$

- Consider the map $f : v \mapsto \int_\Omega x v(x)\,dx$ is an element of X^*

as
- $|\langle f, v\rangle| \leq \|x\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \leq \|v\|_{H^2(\Omega)}$
- linear

So $B : u \mapsto B_0(u) + f$ is well-defined. And therefore also F_μ is well-defined.

▷ WTS: Unique solution

For this, we will use the result from Q1(3). We start by verifying that F_μ is strongly monotone. Consider B as before. We have

$$|\langle B(u) - B(v), u-v\rangle| = |\langle B_0(u-v), u-v\rangle|$$
$$\leq C_3^2 \|u-v\|^2_{H^2(\Omega)}$$

So for $\mu_0 > 0$, chosen st $c_0 - \mu_0 C_3 = c_1 > 0$, we have that
$\forall\ |\mu| \leq \mu_0$

$$\langle F_\mu(u-v), u-v\rangle \geq \underbrace{(c_0 - \mu_0 C_3^2)}_{=c_1} \|u-v\|^2_{H^2(\Omega)}$$

Thus strongly monotone.
The A, B_0 map bounded sets of X to bounded sets of X^*, and, as f is independent of u, F_μ satisfies all assumptions of Q1 (3).
So $\exists!\ u$ st $\langle F_\mu(u), u-v\rangle \leq 0 \quad \forall v \in X$.

Given any w, choosing $v = u \pm w$ gives

$$\langle F_\mu(u), w\rangle \leq 0 \quad \forall w \in X.$$

And therefore $F_\mu(u) = 0$.

(c) $\mu > 0$

$$\langle F_\mu(u), v\rangle = \int_\Omega u'' v''\, dx + \mu(u(0)v(0)) + \mu \int_\Omega x\, v(x)\, dx$$

WTS: $F_\mu(u) = 0 \iff u$ is minimiser of

$$I_\mu(v) := \frac{1}{2}\int_\Omega |v''|^2 dx + \frac{\mu}{2}(v(0))^2 + \mu \int_\Omega x\, v(x)\, dx$$

($\Leftarrow$) u minimiser $\Rightarrow \frac{d}{dt} I_\mu(u+tv)\big|_{t=0} = 0 \quad \forall v$

and this differential is $\langle F_\mu u, v\rangle$

($\Rightarrow$) let v be any element of X, and u st $F_\mu(u) = 0$

Need to show: $I_\mu(v) - I_\mu(u) \geq 0$

Proof:

I_μ is convex:

$$I_\mu(v) - I_\mu(u) = \frac{1}{2}\int_\Omega |v''|^2 - |u''|^2 + \frac{\mu}{2}\left[(v(0))^2 - (u(0))^2\right] + \underbrace{\mu\int_\Omega x(v-u)(x)\,dx}_{*}$$

$$* = -\langle A(u), v-u\rangle - \mu u(0)(v(0) - u(0)) \quad \text{(Since by assumption } F_\mu(u) = 0)$$

So $$I_\mu(v) - I_\mu(u) = \frac{1}{2}\int_\Omega |v''|^2 - 2u'v' + |u''|^2$$

$$+ \frac{\mu}{2}\left[(v(0))^2 - 2v(0)u(0) + (u(0))^2\right]$$

$$= \frac{1}{2}\|(v-u)''\|^2 + \frac{\mu}{2}\left((v-u)(0)\right)^2$$

≥ 0 with equality iff $u = v$.

QUESTION 4. Consider a domain $\Omega \subset \mathbb{R}^n$ which is smooth and bounded, and $g \in C^2(\mathbb{R}^n)$ such that $g \leq 0$ on $\partial\Omega$. Consider the energy I given by

$$I(v) = \int_\Omega |\Delta v|^2 + fv dx,$$

for some $f \in L^2(\Omega)$.

(1) Find the Euler-Lagrange equation satisfied by the critical points of $I(v)$ and prove that every critical point of I is a minimiser.

(2) Consider the set M given by

$$M := \left\{ v \in H^2(\Omega) \cap H_0^1(\Omega) \,|\, v \geq g \text{ a.e. on } \Omega \right\}.$$

Show that there exists a unique minimizer of I on M —check carefully that the assumptions of the Theorem(s) you use are satisfied. You may use without proof that for all $u \in H_0^1(\Omega) \cap H^2(\Omega)$

$$\|u\|_{H_0^1(\Omega)} \leq C \|\Delta u\|_{L^2(\Omega)},$$

where the constant C is independent of u.

(1) i. In order to find the E-L equation, we look for

$$\frac{d}{dt} I(u+tw) = \boxed{2\int_\Omega \Delta u \cdot \Delta v + \int_\Omega fw \overset{!}{=} 0}$$

which is the weak form of E-L eq. We apply ibp and get

$$= \int_\Omega (2\Delta^2 u + f) w \overset{!}{=} 0 \quad \forall w$$

from where we get the strong form of E-L eq:

$$2\Delta^2 u + f = 0$$

ii. Assume u is a critical point so

$$\int 2\Delta u \Delta w + f w = 0 \quad \forall w \in H_0^1(\Omega) \cap H^2(\Omega) \quad (E1)$$

Choose v st $u - v \in H_0^1(\Omega)$ and $v \in H^2(\Omega)$. Then

$$I(v) - I(u) = \int_\Omega |\Delta v|^2 - |\Delta u|^2 + \int_\Omega f(v-u)$$

$$\left(\begin{smallmatrix}\text{use}\\ E1\end{smallmatrix}\right) \quad = \int_\Omega |\Delta v|^2 - |\Delta u|^2 - 2\Delta u \Delta(v-u)$$

$$= \int_\Omega |\Delta v|^2 + |\Delta u|^2 - 2\Delta u \Delta v = \|\Delta(u-v)\|^2_{L^2(\Omega)}$$

≥ 0 with equality iff $\Delta u = \Delta v$.

Since $u = v$ on $\partial\Omega$ (in the sense of traces), the weak max. principle implies that

$u = v$ on Ω (a.e.)

[Alternative would be to use hint of next part for this last bit].

(2) Let $g \in C^2(\mathbb{R}^n)$, $g \leq 0$ on $\partial\Omega$.

$$M = \{ v \in H^2(\Omega) \cap H_0^1(\Omega) : v \geq g \text{ a.e. on } \Omega \}$$

WTS: $\exists$ unique minimiser of I on M.

- Let $X = H_0^1(\Omega) \cap H^2(\Omega)$ with $\|u\|_X^2 = \|u\|^2_{H^1(\Omega)} + \|\Delta u\|_{L^2(\Omega)}$ which is a reflexive separable Banach Space.

We note that M is convex and closed.

- claim: M is non-empty

→ Proof: Let u be the unique solution of

$$\begin{cases} -\Delta u = -\Delta g & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

(which exists because $\Delta g \in C^0(\mathbb{R}^n)$ so $\in L^2(\Omega)$)
Then $-\Delta(u-g)=0$ in Ω. Moreover, $u \geq g$ on $\partial\Omega$
(since by assumption $0 \geq g$ on $\partial\Omega$).
Therefore, by weak max. principle, $u \geq g$ a.e.
and thus $u \in M$.

- Define $A: M \to X^*$ with $A(u)(w) = 2\int_\Omega \Delta u \Delta w + \int_\Omega fw$
$u \mapsto A(u)$

We will use the uniqueness and existence result from Q1.

- Claim: A is continuous and maps bounded sets to bounded sets.

$\to$ Proof: $A(u) = B(u) + F$ for
- $F(v) = \int_\Omega fv\,dx$
- $B(u)(v) = \int_\Omega \Delta u \Delta v\,dx$

As F and $B(u)$ are linear and bounded
$$|F(v)| \leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \|v\|_X,$$

$$|B(u)(v)| \leq \|u\|_{H^2(\Omega)} \|v\|_{H^2(\Omega)},$$

thus we have that $A: M \to X^*$ is continuous.

On the other hand, $u \mapsto B(u)$ is linear and bounded and $\|B(u)\|_{X^*} \leq \|u\|_{H^2(\Omega)}$

From where we get that A maps bounded sets to bounded sets.

- $\langle A(u) - A(v), u-v \rangle = \|\Delta(u-v)\|^2_{L^2(\Omega)}$

Using the hint, we get

$$\geq \frac{1}{2c^2} \|(u-v)\|^2_{H^1(\Omega)} + \frac{1}{2} \|\Delta(u-v)\|^2_{L^2(\Omega)}$$

$$\geq \tilde{C} \|u-v\|^2_X.$$

and thus that A is strictly monotone.

So, existence of a unique solution to

$$\langle A(u), u-v \rangle \leq 0 \qquad (*)$$

follows from Q1.

- It remains to check that this unique solution u is indeed a minimiser of I on M.

Since for every $v \in M$

$$I(v) - I(u) = \int_\Omega |\Delta v - \Delta u|^2 + \langle A(u), v-u \rangle$$

$$\geq 0 \qquad \text{(and equality holds only if } \Delta u = \Delta v)$$

we have that $I(v) \geq I(u) \ \forall v \in M \Rightarrow u$ minimiser of I on M.

Finally, if $\tilde{u}$ were any other minimiser of I on M, we would have that $\tilde{u}$ would also be solution of $(*)$. Thus, $\tilde{u}$ must be equal to u.

QUESTION 5. **Three approaches to the same problem.** Consider a domain $\Omega = \{(x,y) \in \mathbb{R}^2 \text{ s.t.} x^2+y^2 \leq 1\}$ and the equation

$$-\Delta u + u^5 = 1 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

a) • Show that this equation makes sense in $H_0^1(\Omega)$, that is, it has a legitimate weak variational formulation.

b) • Using the first part of the course, show that you can formulate it as a fixed point problem of the form $u = T(u)$ where T is a continuous compact map.

c) • Find a simple subsolution $\underline{u}$ and a simple supersolution $\bar{u}$. Show that the problem can be transformed into

$$-\Delta u + \lambda u = f_\lambda(u)$$

for a constant $\lambda > 0$ chosen so that $f_\lambda(u)$ is increasing when $\underline{u} \leq u \leq \bar{u}$, and use the method of sub and super solutions to show that a solution u can be found by a constructive (iterative) method.

d) • Using Schauder's FPT and the above show that there exists a solution.

e) • Use the variational inequality approach to find a solution in $H_0^1(\Omega)$.

f) • What can you say about uniqueness?

$\Omega \subset \mathbb{R}^2$ (unit disk). $\quad (P1) \begin{cases} -\Delta u + u^5 = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$

a) As $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ $\forall p < \infty$ (cont. and compact embedding), we have in particular that $u^5 \in L^2(\Omega)$ if $u \in H_0^1(\Omega)$.
Then, the weak formulation: Find $u \in H_0^1(\Omega)$ st

$$\int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega u^5 v \, dx = \int_\Omega v \, dx \quad \forall v \in H_0^1(\Omega)$$

is well-defined.

b) (Reformulate as FP Problem)
Let $T(u) := (-\Delta)^{-1}(1-u^5)$. Then

$$T : H_0^1(\Omega) \xrightarrow[\text{cont. by Lemma 28}]{u \mapsto 1-u^5 =: f(u)} L^2(\Omega) \xrightarrow[\text{compact}]{(-\Delta)^{-1}} H_0^1(\Omega)$$

and therefore T is compact.

c) (simple subsolution/ supersolution)

Take $\bar{u} \equiv 1$, then $-\Delta \bar{u} = 0 = f(\bar{u})$ and $\bar{u} \geq 0$ on $\partial\Omega$ ✓

Set $\underline{u} = 0 \Rightarrow -\Delta \underline{u} = 0 \leq f(\underline{u}) = 1$ ✓

$\underline{u} \leq 0$ on $\partial\Omega$ ✓

(Transform problem)

We have that

$$-\Delta u + u^5 = 1 \Leftrightarrow -\Delta u + \lambda u = 1 - u^5 + \lambda u$$

Thus, we set $f_\lambda(u) = 1 - u^5 + \lambda u$.

For f_λ to be increasing on $[0,1]$ (since $0 \equiv \underline{u} \leq u \leq \bar{u} \equiv 1$)

choose e.g. $\lambda = 5$, then $f'_\lambda(u) = 5 - 5u^4 \geq 0$

(Find solution)

constructive method: Define iteratively

$u_0 = \underline{u}$

$u_j = T_\lambda(u_{j-1})$ where $T_\lambda(u) = (-\Delta + \lambda)^{-1}(f_\lambda(u))$

• Claim : (i) $u_j \geq u_{j-1}$ a.e.

(ii) $u_j \leq \bar{u}$

→ Proof of (i): (By induction)

For $j=1$, we have:

• $(-\Delta + \lambda) u_1 = f_\lambda(u_0) = f_\lambda(\underline{u}) \geq (-\Delta + \lambda)(\underline{u})$

as $\underline{u}$ is subsolution.

• $u_1 = 0 \geq \underline{u}$ on $\partial\Omega$

Thus, as weak max. principle holds also for $-\Delta + \lambda$, $\lambda \geq 0$, we have $u_1 \geq \underline{u} = u_0$ a.e.

For $j \geq 1$, assume claim is true for j.

Then, as $u_j \geq u_{j-1}$ and f_λ is monotonic

$$(-\Delta + \lambda)(u_{j+1}) = f_\lambda(u_j) \geq f_j(u_{j-1}) = (-\Delta+\lambda)(u_j) \text{ in } \Omega$$
$$u_{j+1} = u_j \quad \text{on } \partial\Omega$$

So, claim true, again by weak max. principle.

→ Proof of (ii): We know that $u_0 \leq \bar{u}$.

If $u_j \leq u$, we get

$$(-\Delta+\lambda)(u_{j+1}) = f_\lambda(u_j) \leq f_\lambda(\bar{u}) \leq (-\Delta+\lambda)\bar{u} \text{ in } \Omega$$
$$u_{j+1} = 0 \leq \bar{u} \quad \text{on } \partial\Omega$$

$\Rightarrow$ (by weak max. principle) $u_{j+1} \leq \bar{u}$.

• Then, we have a monotone sequence $\underline{u} \leq u_j \leq u_{j+1} \leq \bar{u}$.

• In order to show that this iteration will find a solution, we can argue in different ways:

* Approach 1: Could use T compact from $L^p(\Omega)$ to $H_0^1(\Omega)$ for $p \geq 10$.

(u_j) is bounded in $L^p(\Omega)$. So $u_{j+1} = T(u_j)$ has a convergent subsequence in $H_0^1(\Omega)$.

* Approach 2: Using dominated convergence theorem, we have that $u_j \to u$ in L^p for $p<\infty$

So, using only that $T: L^p(\Omega) \to H_0^1(\Omega)$ is continuous, we get

$$u_{j+1} = T(u_j) \longrightarrow T(u) \quad \text{as } u_j \to u$$

for full sequence.

d) ($\exists$ solution via Schauder's FPT)

- We know that $T: M \to M$
  with $M = \{ u \in H_0^1(\Omega) : \underline{u} \le u \le \bar{u} \} \subset H_0^1(\Omega)$

Since we can apply weak max principle to inequalities, we obtain (as above):

$$\begin{cases} (-\Delta + \lambda)(\underline{u}) \le (-\Delta+\lambda)(u) \le (-\Delta+\lambda)(\bar{u}) & \text{in } \Omega \\ \underline{u} \le u \le \bar{u} & \text{on } \partial\Omega \end{cases}$$

- $T(M)$ is precompact since
  * $M \subset L^{10}(\Omega)$ bounded,
  * $T: L^{10}(\Omega) \xrightarrow[\substack{\text{continuous} \\ \text{maps bounded sets} \\ \text{to bounded sets.}}]{u \mapsto f_\lambda(u)} L^2(\Omega) \xhookrightarrow[\text{compact}]{} H_0^1(\Omega)$

So Schauder's v$\underline{\text{III}}$ applies.

e) (variational inequality approach) $\rightarrow$ use Q1.

Let $A(u)(v) = \int_\Omega \nabla u \cdot \nabla v + \int_\Omega u^5 v - \int_\Omega v \, dx$

Claim: $A : H_0^1(\Omega) \to (H_0^1(\Omega))^* = H^{-1}(\Omega)$ is strictly monotone

$\rightarrow$ Proof

$$\langle A(u) - A(v), u - v \rangle = \|\nabla(u-v)\|^2_{L^2(\Omega)} + \int_\Omega (u^5 - v^5)(u-v)$$

We have that $x \mapsto x^5$ is strictly increasing, so $(u^5 - v^5)(u-v) \geq 0$

Then

$$\langle A(u) - A(v), u - v \rangle \geq \|\nabla(u-v)\|^2_{L^2(\Omega)} \overset{\text{(Poincaré)}}{\geq} c_0 \|u - v\|^2_{H_0^1(\Omega)}$$

Also: A maps bounded sets to bounded sets:

Let $R > 0$, $u \in H_0^1(\Omega)$ st $\|u\|_{H^1(\Omega)} \leq R$. Then, by Sobolev embedding, $\exists$ C st

$$\|u\|_{L^{10}(\Omega)} \leq C \|u\|_{H^1(\Omega)}$$

So, $\forall v \in H_0^1(\Omega)$ st $\|v\|_{H^1(\Omega)} = 1$ and hence $\|v\|_{L^2(\Omega)} \leq 1$

$$|A(u)(v)| \leq \|u^5\|_{L^2} \|v\|_{L^2} \leq \|u\|^5_{L^{10}} \|v\|_{L^2} \leq R^5$$

Hence, existence and uniqueness follows from Q1 for inequality. Finally, usual argument that solution of inequality $\Rightarrow$ solution of equality as we are on a vectorial space, completes the task.

f) (uniqueness)

We have uniqueness!

We can see this from the variational inequality approach in e), or directly:

If u, v are two solutions of (P_1). Then, using as test function $\varphi = u - v$ in the weak formulation from a), we get

$$\int_\Omega (\nabla u - \nabla v)^2 + (u^5 - v^5)(u - v) = 0 \implies u = v.$$