Lie Groups

Section C course Hilary 2019

dancer@maths.ox.ac.uk

Example sheet 4

1. Show that on a compact connected Lie group every left Haar measure is actually bi-invariant.

Note: You may find it helpful to consider the pullback of such a measure μ by right translation, ie the measure defined as follows:

$$R_g^* \mu(U) = \mu(U g^{-1})$$

2. Check the following properties hold for a character χ_V associated to a representation V of a compact Lie group G.

1. $\chi_V(1) = \dim V$
2. χ_V is invariant under conjugation, $\chi_V(hgh^{-1}) = \chi_V(g)$
3. $\chi_V = \chi_W$ for equivalent reps $V \simeq W$
4. $\chi_{V \oplus W}(g) = \chi_V(g) + \chi_W(g)$
5. $\chi_{V \otimes W}(g) = \chi_V(g) \cdot \chi_W(g)$
6. $\chi_{V^*}(g) = \chi_V(g^{-1}) = \overline{\chi_V(g)}$

3. Which of the irreducible representations V_n of $SU(2)$ may be regarded as representations of $SO(3)$?

Recalling which of the V_n have a real structure, deduce that for each natural number n we have a real $(2n+1)$-dimensional representation W_n of $SO(3)$.

Show further that the character of W_n is given by

$$\sum_{k=0}^{2n} e^{i(n-k)t}.$$

4. We have seen that a Lie group admits a left-invariant measure (called *left Haar measure*) that is unique up to scale. In certain situations, eg for compact connected groups, we saw in Question 1 that this is actually *bi-invariant*.

Consider the group Aff_1^+ of affine transformations of $\mathbb{R}$:

$$x \mapsto ax + b \quad : \quad a \in \mathbb{R}_{>0},\ b \in \mathbb{R}.$$

Work out the group law on Aff_1^+ and check it is nonabelian. By considering how $da\, db$ transfroms under left and right translations, find expressions for left and right Haar measures on this group. Deduce that Aff_1^+ has no nontrivial bi-invariant Haar measure.

5. Show that a maximal torus in a compact Lie group is maximal among connected Abelian subgroups.

6. Let B denote the subgroup of $GL(3,\mathbb{C})$ consisting of invertible matrices of the form

$$\begin{pmatrix} \alpha & a & b \\ 0 & \beta & c \\ 0 & 0 & \gamma \end{pmatrix} \quad :$$

Check that B is indeed a subgroup, and that there is a homomorphism ϕ from B onto the *complex torus* $T_{\mathbb{C}} \cong (\mathbb{C}^*)^3$ of diagonal elements of B. Show $\ker \phi$ may be identified with the subgroup U consisting of elements of B with diagonal entries equal to 1.

What are the maximal compact connected subgroups of T, B and U? (no need to give detailed proofs).