

## Geometric Group Theory

### Problem Sheet 2

*The starred exercises are optional.*

1. Let  $\langle S|R \rangle$  be a finite presentation of a group  $G$ .
  - i. Explain how to enumerate all words on  $S$  representing the identity in  $G$ .
  - ii. Explain how to enumerate all finite presentations of  $G$ .
2. Let  $\langle S|R \rangle$  be a finite presentation of a finite group  $G$ . Give an algorithm to solve the word problem for this presentation.
3. Show that if a finitely presented group  $G = \langle S|R \rangle$  has a solvable word problem and the finitely presented group  $H = \langle S'|R' \rangle$  is isomorphic to some subgroup of  $G$  then  $H$  also has a solvable word problem. Note that here we *do not assume* that we are given an injective homomorphism  $f : H \rightarrow G$ .
4. If  $H$  is a finitely generated subgroup of  $G$  then the *membership problem* for  $H$  asks whether there is an algorithm to decide if  $g \in G$  lies in  $H$ . Show that the membership problem is solvable for cyclic subgroups of  $F_n$  (the free group of rank  $n$ ). In other words there is an algorithm such that given  $u, w \in F_n$  decides whether  $u \in \langle w \rangle$ .
5. \* Show that the following presentations are presentations of the trivial group:
  - i)  $\langle a, b, c | aba^{-1} = b^2, bcb^{-1} = c^2, cac^{-1} = a^2 \rangle$
  - ii)  $\langle a, b | a^n = b^{n+1}, aba = bab \rangle$
  - iii)  $\langle a, b | ab^n a^{-1} = b^{n+1}, ba^n b^{-1} = a^{n+1} \rangle$ .
6. An infinite finitely generated group is called just infinite if all its quotients are finite groups. Show that every infinite finitely generated group has a quotient that is just infinite.
7.
  - i. Show that that  $G$  is residually finite if and only if for every  $g \in G$  there is some finite index subgroup  $H$  of  $G$ , such that  $g \notin H$ .
  - ii. Show that if  $G$  has a finite index subgroup which is residually finite then  $G$  itself is residually finite.
8. Let  $G$  be a residually finite group. Show that if  $G$  has finitely many conjugacy classes of elements of finite order then  $G$  has a torsion free finite index subgroup.
9. Give an example of a residually finite group which is not Hopf.

**10.** If  $H$  is a subgroup of the free group  $F_n$  of index  $|F_n : H| = r$  show that  $H$  is a free group of rank  $r(n - 1) + 1$ . (*hint:* look closely at the proof that  $H$  is free).

**11.** If  $g \neq 1$  is an element of  $F_n$  show that the normalizer of  $\langle g \rangle$  in  $F_n$  is a cyclic group. (*hint:* if  $u$  is in the normalizer then  $\langle u, g \rangle$  is free.)

**12.** \* We say that a subgroup  $H$  of  $G$  is *separable* if it is equal to the intersection of all finite index subgroups of  $G$  containing it.

Show that every cyclic subgroup of  $F_n$  (the free group of rank  $n$ ) is separable.

*Hint:* It is enough to show that given  $u, v$  there is a homomorphism  $f$  to a finite group such that  $f(u) \notin \langle v \rangle$ . Imitate now the proof in the notes that  $F_n$  is residually finite.

**13.** Determine the center of the group  $\langle a, b | a^2 = b^3 \rangle$ .

**14.** Show that a finite group  $H$  acting on a tree  $T$  either fixes a vertex of  $T$  or fixes a geometric edge of  $T$  (ie  $H \cdot e \subset \{e, \bar{e}\}$  for some edge  $e$ ). Deduce that any finite subgroup of an amalgam  $A *_C B$  is contained in a conjugate of  $A$  or  $B$ .

**15.** Show that if  $A, B$  are residually finite then  $A * B$  is also residually finite.