Quantum Field Theory

Homework Assignment #2

due week 4, as arranged by your tutor

1. (More or less Peskin 2.1b)

   (a) Construct the energy-momentum tensor for classical electromagnetism, using the action given in the previous HW.

   (b) The usual procedure does not give a symmetric tensor; i.e. $T^{\mu\nu} \neq T^{\nu\mu}$. A symmetric one can be constructed using a tensor of the form $\partial_\lambda K^{\lambda\mu\nu}$, where $K^{\lambda\mu\nu}$ is antisymmetric in its first two indices. Show that $\partial_\lambda K^{\lambda\mu\nu}$ is (four) divergenceless, and so

$$\widehat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu}$$

   is conserved. Show that the same energy and momentum result from $\widehat{T}^{\mu\nu}$ as from $T^{\mu\nu}$.

   (c) Show $\widehat{T}^{\mu\nu}$ is symmetric when

$$K^{\lambda\mu\nu} = F^{\mu\lambda} A^\nu \ .$$

   (d) Find the energy and momentum densities resulting from $\widehat{T}^{\mu\nu}$. Are these the same as those coming from $T^{\mu\nu}$?

2. The purpose of this problem is to quantize the "Schrödinger field theory", a non-Lorentz-invariant field theory that arises in understanding Bose-Einstein condensates of cold atoms, as I will explain later in the class. It also arises as the non-relativistic limit of the Klein-Gordon field theory. The classical field theory has action

$$S = \int dt\, d^3x \left(i\psi^* \partial_t \psi - \frac{1}{2M} \nabla\psi^* \nabla\psi - V_{\text{ext}}(\vec{x})\psi^*\psi \right) \tag{1}$$

   where ψ is a complex field.

   (a) Derive the equations of motion for ψ^* and ψ. You should see that they are Schrödinger equation and its complex conjugate. But despite the use of the same Greek letter, ψ is *not* a wavefunction – it is a field.

   (b) Find the canonical momentum Π conjugate to ψ, and then derive the classical Hamiltonian. By the way, the canonical momentum conjugate to ψ^* vanishes, so we do not quantize it independently (it is a constraint, like a Lagrangian multiplier).

   (c) Now set $V_{\text{ext}} = 0$, and then quantize the theory. Do this by imposing the standard canonical commutation relation on the operators $\hat{\Psi}(\vec{x})$ and $\hat{\Pi}(\vec{x})$, and then rewrite in terms of $\hat{\psi}$ and $\hat{\psi}^*$, using the definition of $\hat{\Pi}$. Then define

$$\hat{\psi}(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} a_{\vec{k}} e^{i\vec{k}\cdot\vec{x}}$$

   and work out the commutation relation $[a_{\vec{k}}, a^\dagger_{\vec{k}'}]$. Then find the quantum Hamiltonian in terms of the $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$.

   (d) How is the ground state of the theory characterised? What are the excited states? What are their energies?

   (e) Show that the theory has a conserved charge like the complex Klein-Gordon theory, and work out its form in the quantum theory in terms of the $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$. What is the physical interpretation of this conserved charge?

3. (More or less Peskin 2.2) The purpose of this problem is to fill in some of the details of the quantization of the complex K-G field theory, and to generalise the symmetry analysis. Write

$$\varphi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} (a_{\vec{k}} e^{-ik\cdot x} + b^\dagger_{\vec{k}} e^{ik\cdot x})$$

(a) Find $\Pi(x)$ and $\Pi^\dagger(x)$ in terms of $a_{\vec{k}}$, $b_{\vec{k}}$, $a^\dagger_{\vec{k}}$ and $b^\dagger_{\vec{k}}$. Define the usual canonical commutation relations for the latter, and them derive $[\Pi(\vec{x}), \varphi(\vec{x'})]$ and the others.

(b) Use these expressions to derive the classical expression for the conserved $U(1)$ charge. Then rewrite it in the quantum theory in terms of $a_{\vec{k}}$, $b_{\vec{k}}$ and their hermitian conjugates.

(c) Consider now the case of two complex Klein-Gordon fields $\varphi_1(x)$, $\varphi_2(x)$ with the same mass. Show that there are now *four* conserved charges, one given by the sum of the $U(1)$ charges, and the other three given by

$$Q^j = \frac{i}{2} \int d^3x \left(\varphi^*_a (\sigma^j)_{ab} \pi^*_b - \pi_a (\sigma^j)_{ab} \varphi_b\right)$$

in the classical theory. The σ^a are the Pauli matrices, and the repeated indices summed over. Show that these three do *not* commute with each other in the quantum theory, but instead have the same commutation relations as the generators of angular momentum (this algebra is known as $su(2)$).

4. The purpose of this problem is to prove several quantities are Lorentz invariant.

(a) First show that

$$\delta(f(x)) = \sum_j \frac{\delta(x - x_j)}{|f'(x_j)|}$$

where the x_j are the points obeying $f(x_j) = 0$.

(b) Show that both

$$\int d^4k \; \delta(k_\mu k^\mu - m^2) \qquad \text{and hence} \quad \int d^3\vec{k} \frac{1}{2w_k}$$

are Lorentz invariant. **Hints**: you don't need the explicit expression of a Lorentz transformation, you just need its definition, and the fact that $det(AB) = det(A)det(B)$.

5. This is a problem in dimensional analysis with $\hbar = c = 1/(4\pi\epsilon_0) = 1$. Give all answers in the number of powers of energy, so that e.g. mass has dimension 1, and time has dimension -1.

(a) What is the dimension of length? Force? Newton's gravitional constant? Momentum? Angular momentum? The electric charge? The electric field? The gauge field A_μ? The action?

(b) Consider a scalar field ϕ in d spacetime dimensions (thus $d-1$ space dimensions). Its Lagrangian density is

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \sum_n a_n \phi^n$$

where ∂_μ is the obvious generalization of a four-vector to a d-vector. What is the dimension of the field ϕ? You can check this by examining the canonical commutation relations. Find the dimensions of the action, the Lagrangian density, and the coefficients a_n. A check on your results is that the dimension of a_2 should always be 2. Why is that?

(c) In $d = 4$, which a_n have positive dimension?