Practical Numerical Analysis: Sheet 1

1. Compute the degree n polynomial interpolant to the function $f(x) = \sin(x) + \sin(x^2)$ on the interval $[0, 4]$ with $n = 4, 8, 16, 32, 64$ using both a monomial and a Lagrange basis combined with both equispaced and Chebyshev nodes.

   Compute the interpolation error for each n and set of nodes by evaluating the interpolant on an equispaced fine grid on $[0, 4]$ and then computing the infinity norm of the difference between the interpolant and the exact values of $f(x)$ at those points. Plot the error values against n for each method and explain the results. (Recall that the Horner scheme can be used for evaluation using the monomial basis and the barycentric formula can be used with the Lagrange basis.)

   For the monomial basis, compute the condition number of the Vandermonde matrix for each point set and plot the results to convince yourself that these matrices can be ill-conditioned.

2. Again consider the function $f(x) = \sin(x) + \sin(x^2)$ on the interval $[0, 4]$ and a grid with $n = 8, 16, 32, 64, 128$ equal mesh spacings. Construct the piecewise linear spline approximation to $f(x)$ and again approximate the infinity norm of the error by using a fine grid. Plot the error values against n and explain the results.

3. Use compactly supported radial basis functions of the form $\phi(r) = (4r + 1)(1 - r)_+^4$ to approximate the so-called Franke function defined by

$$
\begin{aligned}
f(x, y) &= \frac{3}{4}\mathrm{e}^{-((9x-2)^2+(9y-2)^2)/4} + \frac{3}{4}\mathrm{e}^{-((9x+1)^2/49+(9y+1)/10)} \\
&\quad + \frac{1}{2}\mathrm{e}^{-((9x-7)^2+(9y-3)^2)/4} - \frac{1}{5}\mathrm{e}^{-((9x-4)^2+(9y-7)^2)}
\end{aligned}
$$

   in the domain $[0, 1]^2$. (Try `help franke` in Matlab.) Use an equally spaced mesh with $n = 4, 8, 16, 32, 64$ mesh spacings in each direction. For each mesh compute the infinity norm of the error and the condition number of the Vandermonde matrix with scaling factors of $\delta = 0.5, 2, 4/n$.

Further Reading

1. E. Süli and D. Mayers, *An Introduction to Numerical Analysis*, CUP, 2003.
2. L.N. Trefethen, *Approximation Theory and Approximation Practice*, SIAM, 2013.
3. H. Wendland, *Scattered Data Approximation*, Cambridge Monographs on Applied and Computational Mathematics, 2004.