B5.4 consultation session 2

Email questions:

✓ 2013 Q2 (c) (ii)

→ 2016 Q3, characteristic diagram

✓ 2012 Q2 (c) (iii)

* 2012 Q3 (b) (ii), strategy ←

→ 2018 Q1 (c) (iii)

✓ 2015 Q2 (c), streamlines

→ 2016 Q1 (b); Q2 (d) physics; Q3 (b-d)

→ 2016 Q1

→ 2016 Q1 (b), (c), (d); Q2 (c)

✓ 2015 Q3 (c) → see online notes!

✓ 2013 Q3 (b)

→ 2016 Q3 (b, c)

2014 Q3 (d)

$\boxed{\text{B5.4 / 2013 / Q2 (c)}}$

(c)(i) $M = (1+y)^{1/2} \Rightarrow \quad 0 = -y\phi_{xx} + \phi_{yy}$

Hyperbolic for $y > 0$

Elliptic for $y < 0$

(ii) $\phi = \mathrm{Re}\left(e^{ikx} A(y)\right)$

$$\Rightarrow \qquad -k^2 y A = \frac{d^2 A}{dy^2}$$

$$\Rightarrow \qquad \frac{d^2 A}{d\xi^2} - \xi A = 0 \qquad \left(y = -k^{2/3}\xi\right)$$

Hint $\Rightarrow$ A oscillatory for $\xi < 0$, $y > 0$

A decays exponentially for $\xi > 0$, $y < 0$

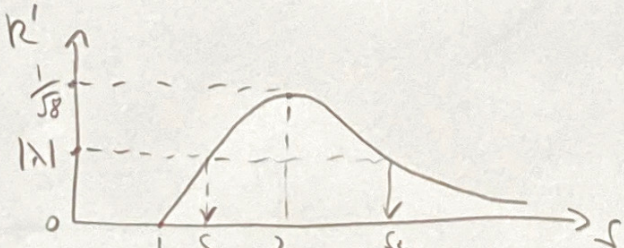
This fits with (c)(i) because disturbances should propagate when equation is hyperbolic but decay when equation is elliptic.

$$\boxed{\text{B5.4 / 2012 / Q2(c)(iii)}}$$

$$m(x, \lambda x) = \int_{-\infty}^{\infty} \frac{\hat{m}_0(l)}{4\pi} \left(e^{i(l\lambda + k(l))x} + e^{i(l\lambda - k(l))x} \right) dl$$

$$\underbrace{\lambda = -k'(l) \qquad\qquad \lambda = k'(l)}_{\text{Dominant contribution from each of these critical wavenumbers.}}$$

Sheet 3, Q1 : $\quad k'(l) = \left(\frac{s-1}{2s^2}\right)^{1/2}, \quad s(l) = \left(1 + \frac{4U^4 l^2}{g^2}\right)^{\frac{1}{2}} \geq 1$



$|\lambda| < \frac{1}{\sqrt{8}} \Rightarrow \exists$ roots $s_1, s_2 > 1$

$\Rightarrow \exists$ roots to $|\lambda| = k'(l)$

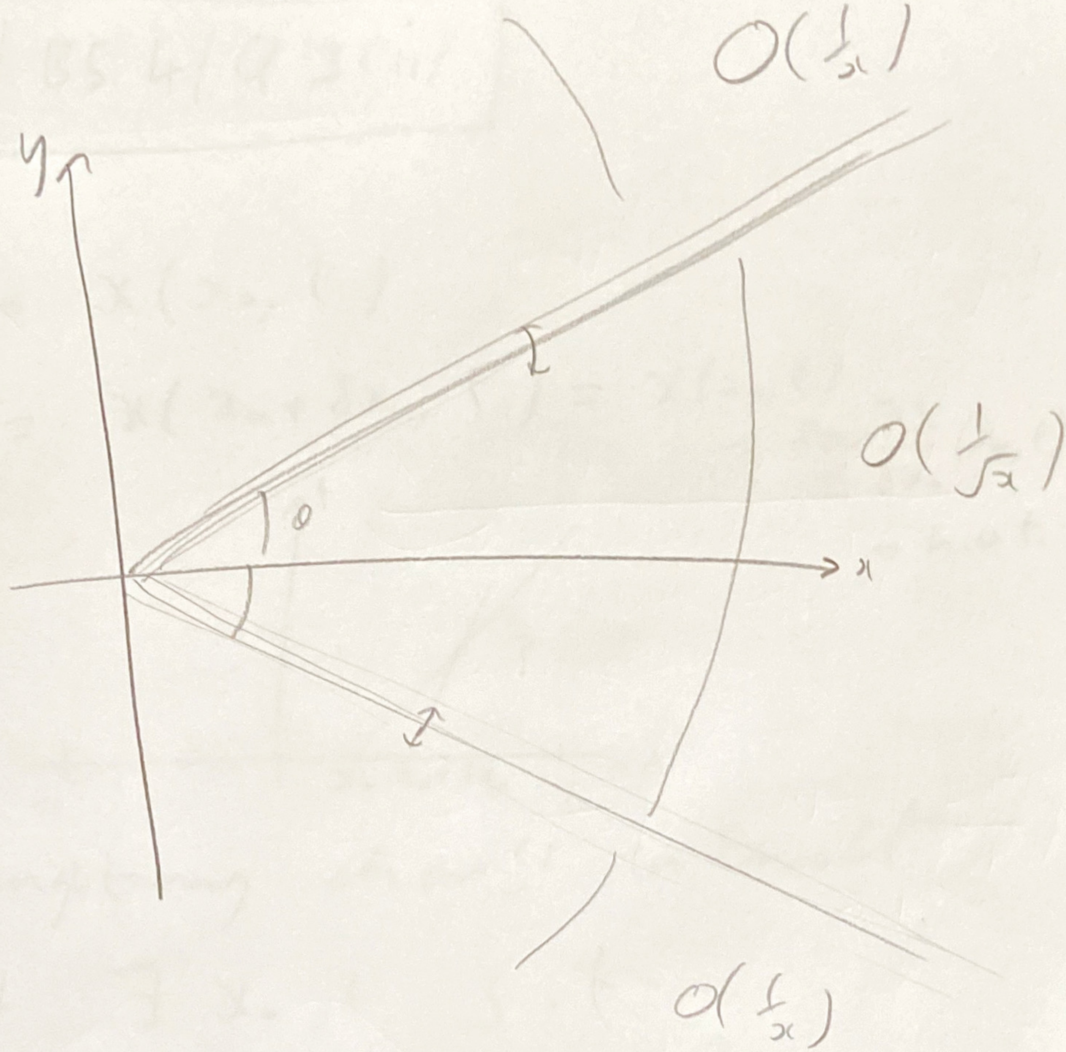
$\Rightarrow m(x, \lambda x) = O\left(\frac{1}{\sqrt{x}}\right)$ as $x \to \infty$

$|\lambda| > \frac{1}{\sqrt{8}} \Rightarrow$ no roots for s

$\Rightarrow$ no points of stationary phase

$\Rightarrow m(x, \lambda x) = O\left(\frac{1}{x}\right)$ as $x \to \infty$

Hence wave pattern "restricted" to $|y| \leq \frac{x}{\sqrt{8}}$ or $|\theta| \leq \sin^{-1}\left(\frac{1}{3}\right)$

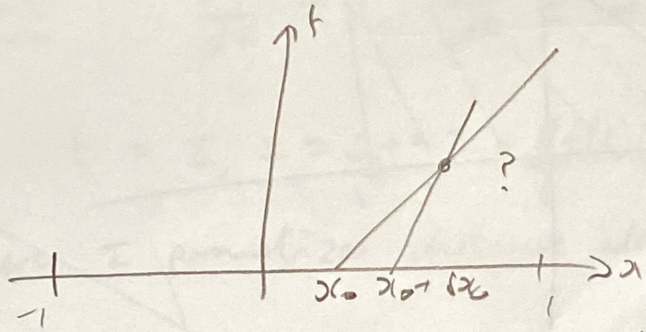


$\cancel{x = x(x_0, (}$

2012/B5.4/Q3(ii)

$x = x(x_0, t)$

$x = x(x_0 + \delta x_0, t) = x(x_0, t) + \delta x_0 \frac{\partial x}{\partial x_0}(x_0, t) + \text{h.o.t.}$



Neighbouring char^cs intersect

if $\exists\, x_0, t > 0$ s.t.

$$\frac{\partial x}{\partial x_0} = 1 + \frac{\beta}{\delta} v'(x_0)(1 - e^{-\delta t}) = 0$$

$$\delta = -\beta v'(x_0)(1 - e^{-\delta t}) \quad (+)$$

If $\delta > \beta \max\limits_{v_0'(x_0) < 0} |v_0'(x_0)|$

then $\delta > \text{RHS}(+)$ $\forall x_0, t > 0$.

$\boxed{\text{B5.4 / 2012 / Q3(c)}}$

Given $v_t + (\alpha + \beta v) v_x = -\delta v$ for $t > 0$

with $x = s$, $t = 0$, $v = f(s)$ for $s \in \mathbb{R}$

<u>Char^{cs}</u> : $\dfrac{dt}{d\tau} = 1$, $\dfrac{dx}{d\tau} = \alpha + \beta v$, $\dfrac{dv}{d\tau} = -\beta v$

$\Rightarrow$ $t = \tau$, $x = s + \alpha\tau + \dfrac{\beta f(s)}{\delta}(1 - e^{-\delta\tau})$, $v = f(s) e^{-\delta\tau}$

where τ parametrizes distance along char^{c} curves.

Jacobian

$$J(s,\tau) = \frac{\partial(x,t)}{\partial(s,\tau)} = \det\begin{pmatrix} x_s & t_s \\ x_\tau & t_\tau \end{pmatrix} = x_s$$

$$\Rightarrow \quad J(s,\tau) = 1 + \frac{\beta}{\delta} f'(s)(1 - e^{-\delta t})$$

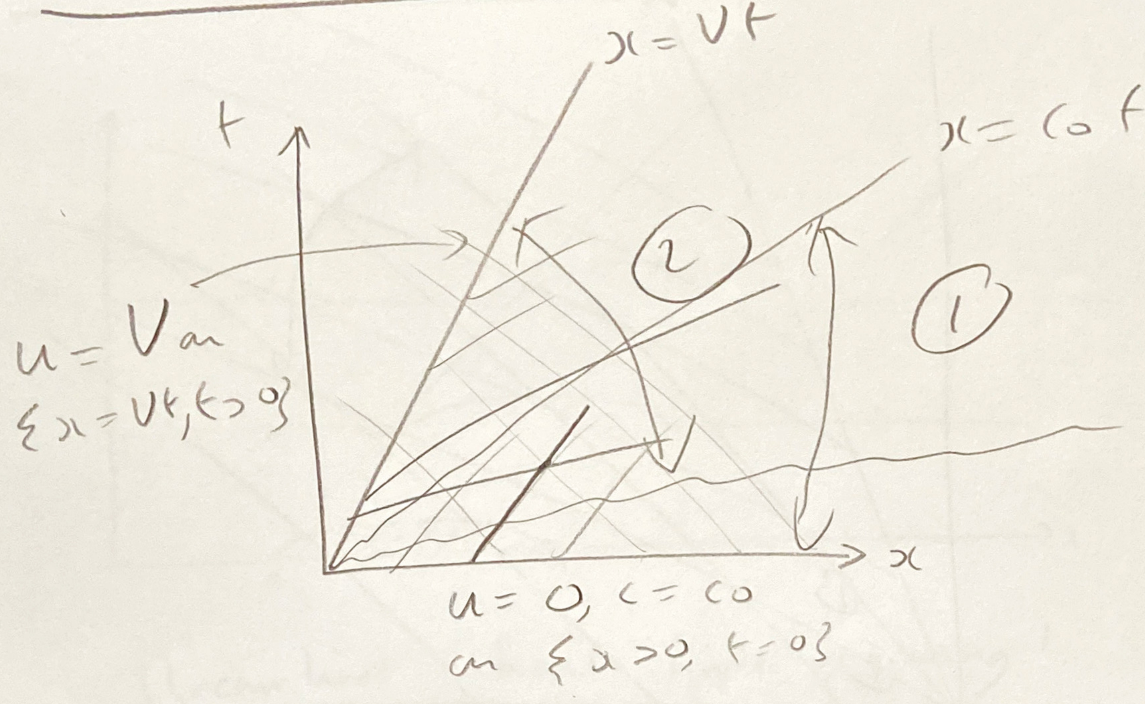
Hence $J = 0$ iff $\delta = -\beta f'(s)(1 - e^{-\delta\tau})$ $(\dagger)$

Hence $\delta > \beta \max\limits_{s: f'(s) < 0} |f'(s)| \Rightarrow \delta > \text{RHS } (\dagger)$ for $s \in \mathbb{R}$, $\tau > 0$

$\Rightarrow J \neq 0$ and no shock can form.

[See Part A DEs I Online Notes)

B5.4 / 2013 / Q3(b)



$(-)$ $u - 2c = -2c_0$ everywhere

$(+)$ from $\{x > 0, t = 0\}$, $u + 2c = 2c_0$

$\Rightarrow \quad u = 0, \ c = c_0$ in $x > c_0 t, t > 0$ (1)

$(+)$ from $\{x = Vt, t > 0\}$, $u + 2c = R_+$

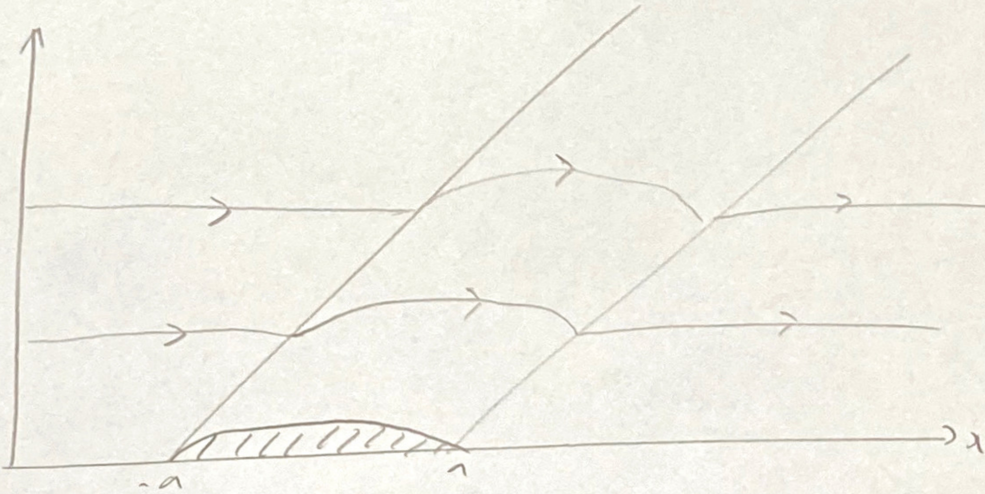
$\Rightarrow$ u, c const $\Rightarrow$ $u = V$, $c = c_0 + \frac{V}{2}$

$\Rightarrow$ $\frac{dx}{dt} = u + c = c_0 + \frac{3}{2}V$ on

+ char^{cs} $\Rightarrow$ maps out

$Vt < x < (c_0 + \frac{3}{2}V)t$ (2)

2015/ B5.4/ Q2(c)



Streamlines inherit shape of wing!

See Acheson.