

B5.4/2016/Q1

(a) B/S - see online notes §3.2, Example 3. NB $c_m = \sqrt{\frac{T}{\sigma}}$

(b) B/S - see online notes §2.2 & 2.3

Note that pressure perturbation $p' = -p_0 \phi_t$ gives additional force per unit area acting on membrane, so must be added to RHS of membrane equation, i.e. $\sigma \omega_{tt} = T \nabla^2 \omega + p'$ on $z = d + \omega$ before linearizing onto $z = d$.

(c) Wave equation $\Rightarrow -\frac{\omega^2}{c_0^2} = \underbrace{\frac{f'' + \frac{1}{r} f'}{f}}_{-\mu^2} + \underbrace{\frac{g''}{g}}_{-\frac{\lambda^2}{a^2}}$

$\phi_z = 0$ on $z = 0 \Rightarrow g(z) = A \cos \frac{\lambda z}{a}$

ϕ bdd at $r=0$ & $\phi_t = 0$ on $r=a \Rightarrow f(r) = (J_0(\frac{\beta_i r}{a}))$, $\mu = \frac{\beta_i}{a}$
as in part (b).

Membrane equation $\Rightarrow -\sigma \omega^2 = -\mu^2 T + p_0 \epsilon \omega A \cos \lambda$

KBC on $z = 0 \Rightarrow -\frac{A \lambda}{a} \sin \lambda = -i \omega$

Eliminate $A \Rightarrow \omega^2 \left(\lambda \sin \lambda - \frac{\rho_0 d}{\sigma} \cos \lambda \right) = c_m^2 \frac{\beta_i^2}{a^2} \lambda \sin \lambda$
 $\uparrow c_0^2 \left(\frac{\beta_i^2}{a^2} + \frac{\lambda^2}{a^2} \right)$

For each i , find countably infinite set $\lambda = \lambda_{ij}$.

(d) $\frac{\rho_0 d}{\sigma} \ll 1 \Rightarrow \lambda \sin \lambda \left(c_0^2 \left(\frac{\beta_i^2}{a^2} + \frac{\lambda^2}{a^2} \right) - c_m^2 \frac{\beta_i^2}{a^2} \right) = 0$

Either $\lambda = j\pi$ ($j \in \mathbb{Z}$) $\Rightarrow \omega^2 = c_0^2 \left(\frac{\beta_i^2}{a^2} + \frac{j^2 \pi^2}{a^2} \right)$

or $\omega^2 = c_0^2 \left(\frac{\beta_i^2}{a^2} + \frac{\lambda^2}{a^2} \right) = c_m^2 \frac{\beta_i^2}{a^2}$

physics: gas light enough that normal modes are uncoupled

B5.4 / 2016 / Q2

(a) B - see online notes §2.3 & sheet 1, Q4

(b) B/s - see online notes §3.4 & sheet 3, Q(ii)

Note $\hat{m}_{tt} + \omega(k)^2 \hat{m} = 0$ for $t > 0$ with
 $\hat{m} = \hat{f}$ and $\hat{m}_t = 0$ at $t = 0$, where $\omega(k) = \sqrt{gk \tanh(kh)}$

$$\Rightarrow F(k) = \hat{f}(k).$$

(c) S/IV - see online notes §3.5, "group velocity" section.

$$\text{Here } n(Vt, t) = I_+(t) + I_-(t)$$

$$\text{where } I_{\pm}(t) = \int_{-\infty}^{\infty} \frac{\hat{f}(k)}{4\pi} e^{i\psi_{\pm}(k)t} dk$$

$$\psi_{\pm}(t) = kV_{\pm} + \omega(k).$$

As explained in motivation to §3.5, dominant contributions to $I_{\pm}(t)$ as $t \rightarrow \infty$ are from neighborhoods of points of stationary phase $k = k_{\pm}^*$ where $\psi'_{\pm}(k_{\pm}^*) = 0$, i.e. from critical wavenumbers $k = k_{\pm}^*$ s.t. $\pm \omega'(k_{\pm}^*) = V$ or $\pm c_g(k_{\pm}^*) = V$.

$$\text{Here, } c_p = \frac{\omega}{k} = \sqrt{\frac{g}{k} \tanh kh}$$

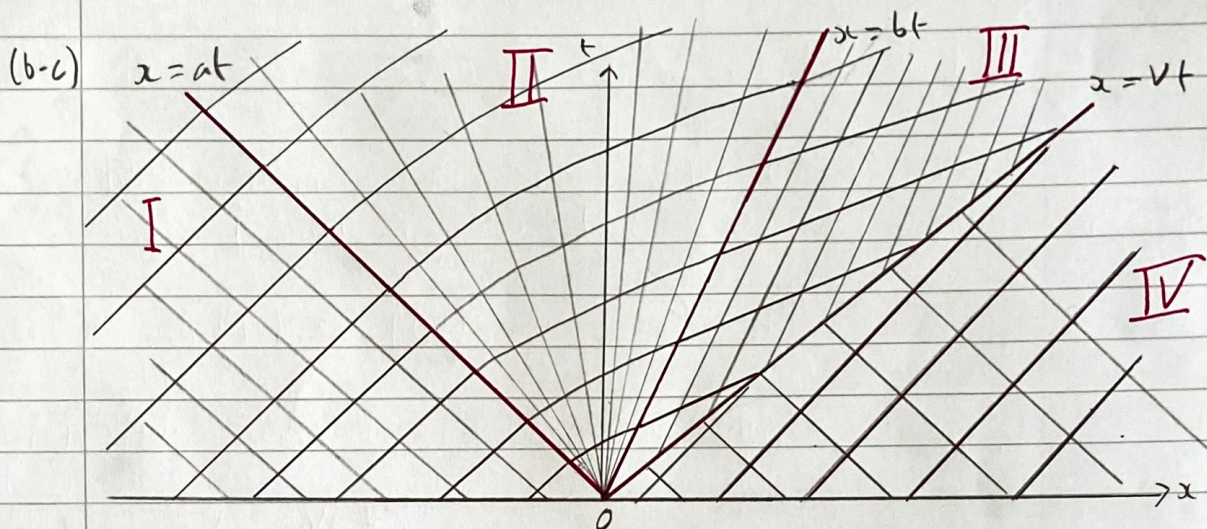
$$c_g = \omega' = \frac{1}{2} c_p \left(1 + \frac{2kh}{\sinh(2kh)} \right)$$

(i) $V \approx \sqrt{gh} \Rightarrow c_g \approx \sqrt{gh} \Rightarrow k_{\pm}^* h \ll 1 \Rightarrow c_p \sim c_g$
 $\Rightarrow$ crests move with wave packet.

(iii) $V \ll \sqrt{gh} \Rightarrow c_g \ll \sqrt{gh} \Rightarrow k_{\pm}^* h \gg 1 \Rightarrow c_p \sim 2c_g$
 $\Rightarrow$ crests move through wave packet as in Fig 3.4(a)

B5.4/2016/Q3

(a) B - see online notes §4.3 & sheet 4, Q1



Region	C_- char ^{cs}	C_+ char ^{cs}
I	From $\{x < 0, t = 0\}$, so $u - 2c_- = -2c_-$	From $\{x < 0, t = 0\}$, so $u + 2c_+ = 2c_+$
II	From $\{x = 0, t = 0\}$, with $u - c = \frac{2}{3}c$	
III	From $\{x = vt, t > 0\}$, with $u - 2c_- = u - 2c_+$	
IV	From $\{x > 0, t = 0\}$, so $u - 2c_- = -2c_-$	

NB: C_- char^{cs} everywhere straight; C_+ char^{cs} straight except in expansion fan region II.

C_+ char^{cs} carry info into LHS shock $\Rightarrow u_+ + 2c_+ = 2c_+$
 C_+ char^{cs} carry info into RHS shock $\Rightarrow u_+ = 0, c_+ = c_R$

RHCS $\Rightarrow V = - \frac{h_- u_-}{h_+ - h_-} = \left(\frac{g(h_+ + h_-)h_-}{2h_+} \right)^{1/2}$
 (with $u_+ = 0$)

→ combo to get expressions for V in part (b); sketch e.g. $V(h_- - h_R)$ as a fn. of h_- two ways to get uniqueness of h_- & hence u_- & V .

Into above $\Rightarrow a = -c_-, b = 2c_- - 3c_+; c = c_+, u = 0$ in I;
 $u = \frac{1}{3}(2c_- - \frac{2}{3}c_+)$, $u = \frac{2}{3}(c_+ - \frac{2}{3}c_+)$ in II; $c = c_+, u = 2(c_- - c_+)$ in III;
 and $c = c_R, u = 0$ in IV.

B5.4/2017/Q1

(a) B - see online notes §2.3 e sheet 1, Q4

(b) Ansatz: $\eta = e^{-i\omega t} x(z)$, $\phi = e^{-i\omega t} x(z) Z(z)$

$$\nabla^2 \phi = 0 \Rightarrow \frac{x''}{x} = -\frac{Z''}{Z} = -k^2 \in \mathbb{R}_0^+$$

as trig solutions are required to satisfy $x'(0) = x'(a) = 0$

$$x'' + k^2 x = 0 \text{ with } x'(0) = x'(a) = 0 \Rightarrow x = A \cos(kx), k = \frac{n\pi}{a} \text{ with } n=1, 2, \dots$$

$$Z'' - k^2 Z = 0 \text{ with } Z'(-h) = 0 \Rightarrow Z = B \cosh k(z+h)$$

Linearized free surface conditions then give the required dispersion relation.

(c) New BC on $x=0, a$ is $\phi_x = \text{Re}(\varepsilon \eta e^{-i\omega t})$

Show problem for $\tilde{\phi}$ with $\omega = \varepsilon \omega$ is the same as for ϕ except for linearized dynamic BC, which becomes

$$\tilde{\phi}_t + g\eta = i\varepsilon \omega^2 (x - \frac{a}{2}) e^{-i\omega t} = i\varepsilon \omega^2 e^{-i\omega t} \sum_{n=1}^{\infty} \frac{4 \cos((2n-1)\frac{\pi z}{a})}{a(2n-1)^2 (\frac{\pi}{a})^2}$$

by the hint. Hence superimpose solns in part (b):

$$\eta = \sum_{n=1}^{\infty} a_n \eta_n, \quad \tilde{\phi} = \sum_{n=1}^{\infty} a_n \phi_n, \text{ but with } \omega = \varepsilon \omega.$$

This (formally) satisfies everything except the inhomogeneous dynamic BC above.

Substituting for η & $\tilde{\phi}$ & equating Fourier coeffs gives the a_n 's and hence the series solution.

Note that the $n=2$ mode dominates for ε close to ω_2 .

BS.4/2017/Q2

(a) B - see online notes §3.5.

(b,c) B/S - see sheet 3, Q1.

Note $\hat{\phi}(x, l, z) = \int_{-\infty}^{\infty} \phi e^{-ily} dy$, $\hat{m}(x, l) = \int_{-\infty}^{\infty} m e^{-ily} dy$

$\Rightarrow \hat{\phi}_{xx} + \hat{\phi}_{zz} = l^2 \phi$ in $z < 0$ with $\hat{\phi}_z = U \hat{m}_x$
and $U \hat{\phi}_x + g \hat{m} = 0$ in $z = 0$; $\hat{\phi}_z \rightarrow 0$ as $z \rightarrow -\infty$
and $\hat{m} = \hat{m}_0$, $\hat{m}_x = 0$ at $x = 0$.

Key step: seek sep. soln $\hat{\phi} = X(x)Z(z)$.

$\Rightarrow \frac{X''}{X} + \frac{Z''}{Z} = l^2 \Rightarrow \frac{X''}{X} = \text{const.} = -k^2$

for oscillatory solutions in the x -direction.

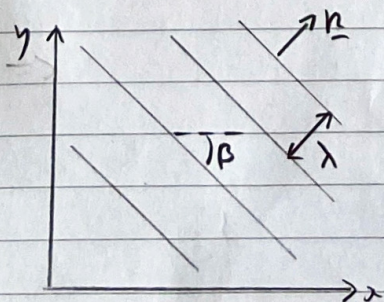
$\Rightarrow \hat{\phi} = (A(l) \cosh kz + B(l) \sinh kz) e^{(k^2 + l^2)^{1/2} z}$

BCs in $z = 0 \Rightarrow \hat{m} = \frac{kU}{g} (A \sinh kz - B \cosh kz)$, $(k^2 + l^2)^{1/2} = \frac{U^2 k^2}{g}$

ILs in $x = 0 \Rightarrow A = 0$, $\hat{m}_0 = -\frac{BkU}{g} \Rightarrow \hat{m} = \hat{m}_0 \cosh(kx)$
and invert.

(d) Waves at edge of wake ($\lambda = \frac{1}{\sqrt{3}}$ say) are those
with $s=2$, i.e. $l = \mp \frac{g}{U^2} \frac{\sqrt{3}}{2}$, $k = \frac{g}{U^2} \frac{\sqrt{3}}{2}$.

Hence, wavenumber vector $\underline{k} = (k, l) = \frac{g}{U^2} \frac{\sqrt{3}}{2} (1, \mp \frac{1}{\sqrt{3}})$



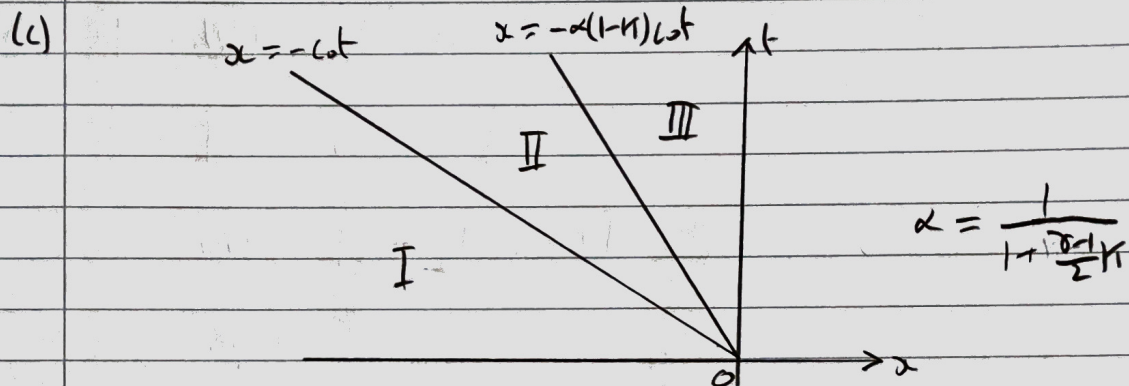
Wavecrests are $\underline{k} \cdot (x, y) = \text{const}$
as illustrated for + sign.

$\Rightarrow \cos \beta = \frac{1}{\sqrt{3}}$, $\lambda = \frac{2\pi}{|k|} = \frac{4\pi U^2}{3g}$

B5.4/2017/Q3

(a) B - see online notes §1.9 and sheet 0, Q3

(b) B - see online notes §4.2 & sheet 3, Q3



($+$ char^U on which $u + \frac{2c}{\delta-1} = \frac{2c_0}{\delta-1}$ come everywhere from $\{x < 0, t = 0\}$.

I: ($-$ char^U on which $u - \frac{2c}{\delta-1} = -\frac{2c_0}{\delta-1}$ come from $\{x < 0, t = 0\} \Rightarrow u = 0, c = c_0$.

III: ($-$ char^U on which $u - \frac{2c}{\delta-1} = \text{const}$ come from $\{x = 0, t > 0\}$ on which $u = \kappa c \Rightarrow c = \alpha c_0, u = \alpha \kappa c_0$

II: Needed because $-ct < -\alpha(1-\kappa)ct < 0$ for $t > 0$ because $0 < \alpha < 1$ and $0 < \kappa < 1$. Now ($-$ char^U come from origin (expansion fan)

$$\Rightarrow c = \frac{2}{\delta+1} \left(c_0 - \frac{\delta-1}{2} \frac{x}{t} \right), u = \frac{2}{\delta+1} \left(c_0 + \frac{x}{t} \right)$$

135.4/2018/Q1

(a) B - see online notes § 2.2

(b) S - see § 2.2 "Waves due to a point source."

$$\cdot r\phi(r,t) = f(r)e^{-i\omega t}$$

$$\Rightarrow f'' + \frac{\omega^2}{c_0^2} f \quad \text{for } 0 < r < a$$

$$\text{with } f(0) = 0 \text{ (so } |\phi(0,t)| < \infty \text{)} \text{ and } \left. \frac{d}{dr} \left(\frac{f}{r} \right) \right|_{r=a} = 0$$

$$\Rightarrow f(r) \propto \sin\left(\frac{\omega r}{c_0}\right), \quad \tan\left(\frac{\omega a}{c_0}\right) = \frac{\omega a}{c_0}$$

(c) (i) Linearizing $\Rightarrow \phi_r = -i\omega a \xi e^{-i\omega t}$ on $r = a$ (†)

$$\text{(ii) } r\phi(r,t) = f(r)e^{-i\omega t} \Rightarrow f(r) = C e^{i\omega r/c_0} + D e^{-i\omega r/c_0} \quad (C, D \in \mathbb{C})$$

$$\Rightarrow \phi = \underbrace{\frac{C}{r} e^{\frac{i\omega}{c_0}(r-ct)}}_{\text{outward}} + \underbrace{\frac{D}{r} e^{-\frac{i\omega}{c_0}(r+ct)}}_{\text{Inward}}$$

Radiation condition $\Rightarrow D = 0$, so C given by (†)

$$\text{(iii) For } \omega \neq \frac{c_0 \alpha_n}{a} \quad \forall n, \quad \phi = \frac{A}{r} \sin\left(\frac{\omega r}{c_0}\right) e^{-i\omega t}$$

with A given by (†).

For $\omega = \frac{c_0 \alpha_m}{a}$ for some m , try instead

$$\text{secular solution } \phi = \left(\frac{f(r)}{r} + \frac{g(r)}{r} \right) e^{-i\omega t}$$

$$\Rightarrow g'' + \frac{\omega^2}{c_0^2} g = 0 \quad f'' + \frac{\omega^2}{c_0^2} f = -\frac{2i\omega}{c_0^2} g$$

for $0 < r < a$, with $|\phi(0,t)| < \infty$ giving BCs

BS.4/2018/Q2

(a) B - see online notes §3.5 & sheet 3, Q(1)

(b) Fourier transform in $x \Rightarrow$

$$(ik)^2 \hat{\phi} + \hat{\phi}_{zz} = 0 \quad \text{for } z < 0$$

$$\hat{\phi}_z = \hat{u}_t, \quad \rho \hat{\phi}_t + B(ik)^4 \hat{u} = 0 \quad \text{on } z = 0$$

$$\hat{\phi} \rightarrow 0 \quad \text{as } z \rightarrow -\infty$$

$$\hat{u} = 0, \quad \hat{u}_t = - \int_a^x W e^{-ikx} dx = -\frac{2W}{k} \sin(ka) \quad \text{at } t=0$$

$$\text{Hence, } \hat{\phi} = A(k, t) e^{k|z|}, \quad \text{so } |k|A = \hat{u}_t, \quad \rho A_t = -Bk^4 \hat{u}$$

$$\Rightarrow \hat{u}_{tt} = -\frac{B}{\rho} k^4 |k| \hat{u} \quad \text{for } t > 0$$

$$\Rightarrow \hat{u} = -2W \frac{\sin(ka)}{k} \frac{\sin(\omega(k)t)}{\omega(k)}, \quad \omega(k) = \sqrt{\frac{B}{\rho}} |k|^{\frac{5}{2}}$$

Finally, invert for answer.

$$(c) \quad u_t(V, t) = I_+(t) + I_-(t), \quad I_{\pm}(t) = \int_{-\infty}^{\infty} f(k) e^{i\psi_{\pm}(t)k} dk$$

$$\text{where } f(k) = -\frac{W \sin(ka)}{2\pi k}, \quad \psi_{\pm}(t) = kV \mp \omega(k).$$

$$\text{Now apply part (a): } \psi'_{\pm}(k_{\pm}^{\pm}) = 0 \quad \text{iff } k_{\pm}^{\pm} = \pm \left(\frac{2V}{5\sigma} \right)^{2/3},$$

i.e. $\exists!$ point of stationary phase for $I_+(t)$ & $I_-(t)$.

$$\text{Use part (a) with } \omega'(k) = \frac{5}{2} \sigma |k|^{3/2}, \quad \omega''(k) = \frac{15}{4} \sigma |k|^{1/2}$$

$$\text{gives answer with } A = -\frac{W \sin(k_{\pm}^{\pm} a)}{\pi k_{\pm}^{\pm}} \left(\frac{2\pi}{\frac{15}{4} \sigma (k_{\pm}^{\pm})^{1/2}} \right)^{1/2}$$

B5.4/2018/Q3

(a) B/S - see online notes §5.3 & sheet 4, Q3.

Rankine-Hugoniot conditions with $h_- = \alpha h_0$, $u_- = U$, $h_+ = h_0$ and $u_+ = 0$ imply

$$U = \left(1 - \frac{1}{\alpha}\right)V, \quad V = \left(\frac{\alpha(1+\alpha)gh_0}{2}\right)^{1/2},$$

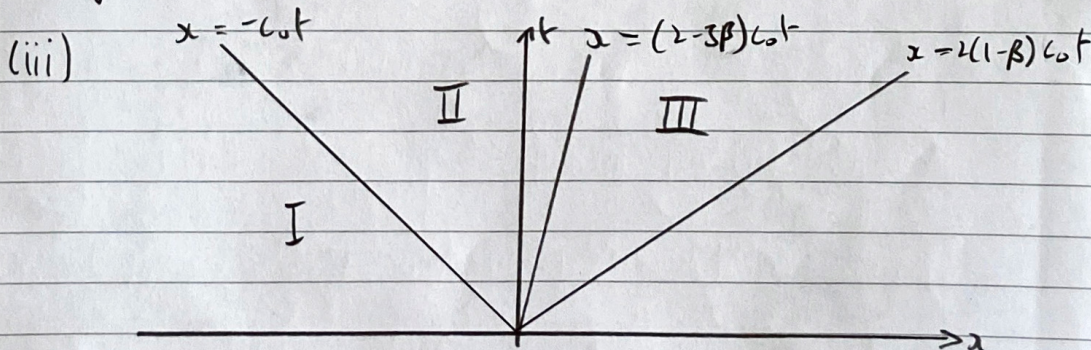
taking the +ve root because $V > U > 0$.

Eliminate V for equation for α .

(b) (i) B - see online notes §4.3 & sheet 4, Q1.

(ii) At $x = X(t)$, $u + 2c = 2c_0 = 2\sqrt{gh_0}$, $u = \dot{X}$
and $\dot{X} = \lambda h^2 = \lambda c^4/g$ by C -char^s, KBC & slip B.

Together these imply $\frac{\lambda c^4}{g} + 2c = 2c_0$ at $x = X$,
so that c is a constant, βc_0 say on dam,
from which the results follow.



C -char^s come everywhere from $\{x < 0, t = 0\}$.

I: C -char^s from $\{x < 0, t = 0\} \Rightarrow u = 0, c = c_0$ there.

III: C -char^s from dam $\Rightarrow u = 2(1-\beta)c_0, c = \beta c_0$ there.

II: Needed because $0 < \beta < 1 \Rightarrow -c_0 t < (2-3\beta)c_0 t < 2(1-\beta)c_0 t$.

I: C -char^s from origin (expⁿ fan) $\Rightarrow u = \frac{2}{3}(c_0 + \frac{x}{t})$
 $c = \frac{1}{3}(2c_0 - \frac{x}{t})$