

2019/BS.4/Q1

(a) Euler equations: $\rho_t + \nabla \cdot (\rho \underline{u}) = 0$, $\rho(\underline{u}_t + (\underline{u} \cdot \nabla) \underline{u}) = -\nabla p$.

Perturb $\rho = \rho_0 + \rho'$, $p = P(\rho_0) + p'$, $\underline{u} = \underline{u}'$, where primed variables are small.

① $\Rightarrow (\rho_0 + \rho')_t + \nabla \cdot (\rho_0 \underline{u}') + \nabla \cdot (\rho' \underline{u}') = 0$
 $\Rightarrow \rho'_t + \rho_0 \nabla \cdot \underline{u}' = 0$ linearizing.

② $\Rightarrow \rho_0 \underline{u}'_t + \rho' \underline{u}'_t + (\rho_0 + \rho') (\underline{u}' \cdot \nabla) \underline{u}' = -\nabla p'$
 $\Rightarrow \rho_0 \underline{u}'_t = -\nabla p'$ linearizing.

$p = P(\rho_0) \Rightarrow P(\rho_0) + p' = P(\rho_0 + \rho') = P(\rho_0) + \rho' c_0^2 + \dots$
by Taylor's Theorem with $c_0 = \left(\frac{dp}{d\rho}(\rho_0)\right)^{1/2}$
 $\Rightarrow p' = c_0^2 \rho'$ linearizing.

[B4]

$\nabla \wedge \text{④} \Rightarrow (\nabla \wedge \underline{u}')_t = -\frac{1}{\rho_0} \nabla \wedge \nabla p' = 0$
 $\Rightarrow \nabla \wedge \underline{u}' = 0$ for $t \geq 0$ since flow irrotational initially
 $\Rightarrow \exists \phi$ s.t. $\underline{u}' = \nabla \phi$ by hint.

④ & ⑥ $\Rightarrow \nabla(p' + \rho_0 \phi_t) = 0$
 $\Rightarrow p' + \rho_0 \phi_t = F(t) = 0$ (wlog absorbing arb. fns F into ϕ)

Hence, $\nabla^2 \phi = -\frac{1}{\rho_0} \rho'_t$ (by ③ & ⑥)
 $= -\frac{1}{\rho_0 c_0^2} p'_t$ (by ⑤)
 $= -\frac{1}{\rho_0 c_0^2} (-\rho_0 \phi_{tt})$ (by ②)

[B4]

giving the wave equation $\phi_{tt} = c_0^2 \nabla^2 \phi$.

□

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(b) [B1] BCS: $\phi_x = 0$ on $x = 0, a$ for $0 < y < b, 0 < z < h$;
 $\phi_y = 0$ on $y = 0, b$ for $0 < x < a, 0 < z < h$;
 $\phi_z = 0$ on $z = 0, h$ for $0 < x < a, 0 < y < b$.

Seek nontrivial separable solution $\phi = e^{-i\omega t} F(x)G(y)H(z)$, then

$$-\frac{\omega^2}{c_0^2} = \underbrace{\frac{F''}{F}}_{-\lambda^2} + \underbrace{\frac{G''}{G}}_{-m^2} + \underbrace{\frac{H''}{H}}_{-\omega^2} \quad (FGH \neq 0)$$

for $\lambda, m, \omega \in \mathbb{R}$ since these quantities must be incl. x, y, z , and signs chosen for nontrivial solutions as BCS \Rightarrow
 $F'(0) = F'(a) = 0, G'(0) = G'(b) = 0, H'(0) = H'(h) = 0$.

[S3]

For $\lambda = 0, F = \text{const}$, but for $\lambda \neq 0, F = A \cos(\lambda x) + B \sin(\lambda x)$ ($A, B \in \mathbb{C}$ arb.), so BCS $\Rightarrow B = 0$ and $\lambda a = m\pi, m \in \mathbb{Z} \setminus \{0\}$.

Combo $\Rightarrow F \propto \cos\left(\frac{m\pi x}{a}\right), \lambda = \frac{m\pi}{a}, m \in \mathbb{Z}$.

Similarly for G and H with $n = \frac{n\pi}{b}, \omega = \frac{k\pi}{h}, n, k \in \mathbb{Z}$.

Combo $\Rightarrow \phi = C e^{-i\omega t} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \cos\left(\frac{k\pi z}{h}\right) \quad (C \in \mathbb{C})$

are normal modes with natural frequencies ω s.t.

$$\omega^2 = c_0^2 (\lambda^2 + m^2 + \omega^2) = \pi^2 c_0^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} + \frac{k^2}{h^2} \right),$$

[S5]

where m, n and k are integers. □

[7]

(c) To satisfy no-flux BCS on walls in $z > 0$ and on waveplate at $z = 0$, seek separable solution

$$\phi = e^{-i\omega t} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) f(z)$$

$$\Rightarrow -\omega^2 f = c_0^2 \left(-\frac{m^2 \pi^2}{a^2} f - \frac{n^2 \pi^2}{b^2} f + f'' \right).$$

[S/N2] $\Rightarrow f'' + \frac{\omega^2 - \omega_{m,n}^2}{c_0^2} f = 0$ for $z > 0$, with $f'(0) = V$

(ii) $\omega > \omega_{m,n} \Rightarrow f = A e^{i\kappa z} + B e^{-i\kappa z}$ ($A, B \in \mathbb{C}$ arb.), $\kappa = \left(\frac{\omega^2 - \omega_{m,n}^2}{c_0^2}\right)^{1/2}$

$\Rightarrow \phi = \left(\underbrace{A e^{i(\kappa z - \omega t)}}_{\text{outward}} + \underbrace{B e^{-i(\kappa z + \omega t)}}_{\text{Inward}} \right) \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$

Radiation condition \Rightarrow no inward travelling waves as $z \rightarrow \infty$, so $B = 0$. Then $f'(0) = V \Rightarrow i\kappa A = V$, giving

[S/N2] $\phi = \frac{V}{i\kappa} e^{i(\kappa z - \omega t)} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$

(swap!)

(i), $0 < \omega < \omega_{m,n} \Rightarrow f = A e^{-\beta z} + B e^{\beta z}$ ($A, B \in \mathbb{C}$ arb.), $\beta = \left(\frac{\omega_{m,n}^2 - \omega^2}{c_0^2}\right)^{1/2}$

ϕ bdd as $z \rightarrow \infty \Rightarrow B = 0$. Then $f'(0) = V \Rightarrow -\beta A = V$, giving

$\phi = -\frac{V}{\beta} e^{-\beta z - i\omega t} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$ □

[S/N2]

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(d) Hint $\Rightarrow \phi_2(x, y, 0, t) = \frac{qV e^{-i\omega t}}{\pi^2} \sum_{m,n=1}^{\infty} \cos\left(\frac{2m\pi x}{a}\right) \cos\left(\frac{2n\pi y}{b}\right)$

for $0 < x < a$, $0 < y < b$, so superimposing the solutions in part (c) gives

$\phi = \frac{qV e^{-i\omega t}}{\pi^2} \sum_{m,n=1}^{\infty} F_{2m,2n}(z) \cos\left(\frac{2m\pi x}{a}\right) \cos\left(\frac{2n\pi y}{b}\right)$

where

$F_{m,n}(z) = \frac{c_0}{i(\omega^2 - \omega_{m,n}^2)^{1/2}} e^{i(\omega^2 - \omega_{m,n}^2)^{1/2} z/c_0}$ for $\omega_{m,n} < \omega$,

$F_{m,n}(z) = -\frac{c_0}{(\omega_{m,n}^2 - \omega^2)^{1/2}} e^{-(\omega_{m,n}^2 - \omega^2)^{1/2} z/c_0}$ for $\omega_{m,n} > \omega$.

[N4]

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□

2019/B5.4/Q2

(a) [B2] Constant density so $\nabla \cdot \mathbf{u} = 0$ and velocity potential is ϕ so $\mathbf{u} = \nabla \phi$, so $\nabla^2 \phi = 0$ in $z < 0$ after linearizing for small m .

[B2] Since $\phi_2(x, m, t) = \phi_2(x, 0, t) + m \phi_{22}(x, 0, t) + \text{h.o.t.}$ by Taylor's Theorem for small m , linearizing and imposing KBC on $z = 0 \Rightarrow \phi_z = \eta_t$ on $z = 0$.

Bernoulli's equation for irrotational flow states $\phi_t + \frac{1}{2} |\nabla \phi|^2 + gz + p/\rho = F(t)$ an arbitrary fn of t . Set $F(t) = 0$ wlog, then DBE becomes

$$\phi_t + \frac{1}{2} |\nabla \phi|^2 + gm + \frac{\epsilon P_a}{\rho(\alpha^2 + \epsilon^2)} = 0 \text{ on } z = \eta.$$

[S/N2] Linearize and impose on $z = 0 \Rightarrow \phi_t + gm = -\frac{\epsilon P_a}{\rho(\alpha^2 + \epsilon^2)}$ on $z = 0$.

[B1] Finally, no flow at $\infty \Rightarrow \phi \rightarrow 0$ as $x^2 + z^2 \rightarrow \infty$. \square

[7]

(b) Let $\hat{\phi}(k, z, t) = \int_{-\infty}^{\infty} \phi(x, z, t) e^{-ikx} dx$, $\hat{\eta}(k, t) = \int_{-\infty}^{\infty} \eta(x, t) e^{-ikx} dx$

Then IBVP in part (a) implies, by hint,

$$\textcircled{1} -k^2 \hat{\phi} + \hat{\phi}_{zz} = 0 \text{ in } z < 0;$$

$$\textcircled{2} \hat{\phi}_z = \hat{\eta}_t, \hat{\phi}_t + g\hat{\eta} = -\frac{P_a}{\rho} e^{-\epsilon|k|} \text{ on } z = 0;$$

$$\textcircled{3} \hat{\phi} \rightarrow 0 \text{ as } z \rightarrow -\infty;$$

$$\textcircled{4} \hat{\eta}(k, 0) = \hat{\eta}_t(k, 0) = 0.$$

[S/B4]

The general solution of ① is

$$\hat{\phi}(r, z, t) = A(r, t) e^{|k|z} + B(r, t) e^{-|k|z},$$

where A, B are arbitrary.

$$\textcircled{2} \Rightarrow B = 0$$

$$\textcircled{2} \Rightarrow |k|A = \hat{m}_t, \quad A_t + g\hat{m} = -\frac{P_a}{\rho} e^{-\varepsilon|k|}$$

$$\Rightarrow \hat{m}_t + g|k|\hat{m} = -\frac{P_a}{\rho} |k| e^{-\varepsilon|k|}$$

$$\Rightarrow \hat{m} = C(r) \cos(\omega(r)t) + D(r) \sin(\omega(r)t) - \frac{P_a}{\rho g} e^{-\varepsilon|k|}$$

where C, D are arbitrary and $\omega(r) = \sqrt{g|k|}$

$$\textcircled{4} \Rightarrow 0 = C(r) - \frac{P_a}{\rho g} e^{-\varepsilon|k|}, \quad 0 = D(r)$$

$$[\varepsilon/N4] \Rightarrow \hat{m}(r, t) = \frac{P_a}{\rho g} e^{-\varepsilon|k|} (\cos(\omega(r)t) - 1)$$

Inverting then gives

$$m(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{m}(k, t) e^{ikx} dk$$

$$= \frac{a}{2\pi} \int_{-\infty}^{\infty} e^{-\varepsilon|k|} (\cos(\omega(r)t) - 1) e^{ikx} dk$$

$$[S1] \text{ where } a = \frac{P_a}{\rho g}$$

[9]

(c) By the hint in part (b), $\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\varepsilon|k|} e^{ikx} dk = \frac{\varepsilon}{\pi(x^2 + \varepsilon^2)}$, so

$$m(x, t) = \frac{-\varepsilon a}{\pi(x^2 + \varepsilon^2)} + \frac{a}{2\pi} \int_{-\infty}^{\infty} e^{-\varepsilon|k|} \frac{1}{2} \left(e^{-i\omega(r)t} + e^{i\omega(r)t} \right) e^{ikx} dk$$

$$\Rightarrow n(vt, t) = -\frac{\varepsilon a}{\pi(v^2 t^2 - \varepsilon^2)} + \int_{-\infty}^{\infty} f(k) \left(e^{i(kv - w(k))t} + e^{i(kv + w(k))t} \right) dk$$

[S/N2]

where $f(k) = \frac{ae^{-\varepsilon|k|}}{4\pi}$

Thus, $\psi(k) = kv \mp w(k)$ in integral in hint.

Note $w(k) = g^{1/2}|k|^{1/2} \Rightarrow \frac{dw}{dk} = \frac{1}{2}g^{1/2}|k|^{-1/2} \text{sgn}(k)$

$$\Rightarrow \frac{d^2w}{dk^2} = -\frac{1}{4}g^{1/2}|k|^{-3/2} \quad (k \neq 0)$$

Thus $\psi'(k) = v \mp \frac{dw}{dk} = v \mp \frac{1}{2}g^{1/2}|k|^{-1/2} \text{sgn}(k)$, so

$$\psi'(k_*) = 0, v > 0 \Rightarrow k_* = \pm \frac{g}{4v^2}$$

Calculate $f(k_*) = \frac{a}{4\pi} e^{-\varepsilon g/4v^2}$

$$\psi(k_*) = \left(\pm \frac{g}{4v^2} \right) v \mp g^{1/2} \left(\frac{g}{4v^2} \right)^{1/2} = \mp \frac{g}{4v}$$

[S/B4]

$$\psi''(k_*) = \mp \frac{d^2w}{dk^2}(k_*) = \mp \frac{g^{1/2}}{4} \left(\frac{4v^2}{g} \right)^{3/2} = \pm \frac{2v^3}{g}$$

Hence, hint gives, as $t \rightarrow \infty$,

$$n(vt, t) + \frac{\varepsilon a}{\pi(v^2 t^2 - \varepsilon^2)} \sim f(k_*) \left(\exp\left(-\frac{igt}{4v} + \frac{igt}{4}\right) + \exp\left(\frac{igt}{4v} - \frac{igt}{4}\right) \right) \times \left(\frac{2\pi}{|2v^3/g|t} \right)^{1/2}$$

Since $\frac{\varepsilon a}{\pi(v^2 t^2 - \varepsilon^2)} \ll t^{-1/2}$ as $t \rightarrow \infty$, we obtain

$$n(vt, t) \sim \frac{A}{t^{1/2}} \cos\left(\frac{gt}{4v} - \frac{\pi}{4}\right)$$

[N/S3]

as $t \rightarrow \infty$, where $A = \frac{pa}{2\pi(\pi g v^3)^{1/2}} \exp\left(-\frac{\varepsilon g}{4v^2}\right)$

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2019/BS.4/Q3

(a) (i) Manipulate $\frac{Dp}{Dt} = -\rho u_x$ ①, $\frac{Du}{Dt} = -\frac{1}{\rho} B_x$ ②, $\rho c_v \frac{DT}{Dt} = -p u_x$ ③, $p = \rho RT$ ④

$$\rho^\gamma \frac{D}{Dt} \left(\frac{p}{\rho^\gamma} \right) = \frac{Dp}{Dt} - \frac{\gamma p}{\rho} \frac{D\rho}{Dt}$$

$$= R \rho \frac{DT}{Dt} + RT \frac{D\rho}{Dt} - \gamma RT \frac{D\rho}{Dt} \quad (\text{by } ④)$$

$$= -\frac{R p u_x}{c_v} + (\gamma - 1) \rho RT u_x \quad (\text{by } ③ \text{ \& } ①)$$

$$= -(\gamma - 1) p u_x + (\gamma - 1) p u_x \quad (\gamma - 1 = \frac{R}{c_v} \text{ \& } ① \text{ again})$$

$\Rightarrow \frac{D}{Dt} \left(\frac{p}{\rho^\gamma} \right) = 0$ so p/ρ^γ is conserved following the flow $\Rightarrow p/\rho^\gamma = p_0/\rho_0^\gamma$ since uniform initially. \square

[B4]

(ii) $p = k\rho^\gamma \Rightarrow c^2 = \frac{\partial p}{\partial \rho} = \gamma k \rho^{\gamma-1}$

$$\Rightarrow \rho = A c^{\frac{2}{\gamma-1}}, \quad p = \frac{A}{\gamma} c^{\frac{2\gamma}{\gamma-1}}, \quad \text{where } A = (\gamma R)^{-\frac{1}{\gamma-1}}$$

$$\Rightarrow p_t = \frac{2}{\gamma-1} \frac{\rho c_t}{c}, \quad p_x = \frac{2}{\gamma-1} \frac{\rho c_x}{c}, \quad p_{xx} = \frac{2\gamma}{\gamma-1} \frac{\rho c_x^2}{c} = \frac{2}{\gamma-1} \rho c c_{xx}$$

Hence, $p_t + u p_x + p u_x = 0 \Rightarrow \frac{2}{\gamma-1} (c_t + u c_x) + c u_x = 0$ ⑤

$$u_t + u u_x + \frac{1}{\rho} B_x = 0 \Rightarrow u_t + u u_x + \frac{2}{\gamma-1} c c_x = 0$$
 ⑥

$$\text{⑤} \pm \text{⑥} \Rightarrow \frac{\partial}{\partial t} \left(u \pm \frac{2c}{\gamma-1} \right) + (u \pm c) \frac{\partial}{\partial x} \left(u \pm \frac{2c}{\gamma-1} \right) = 0$$

Hence, the Riemann invariants $R_\pm = u \pm \frac{2c}{\gamma-1}$

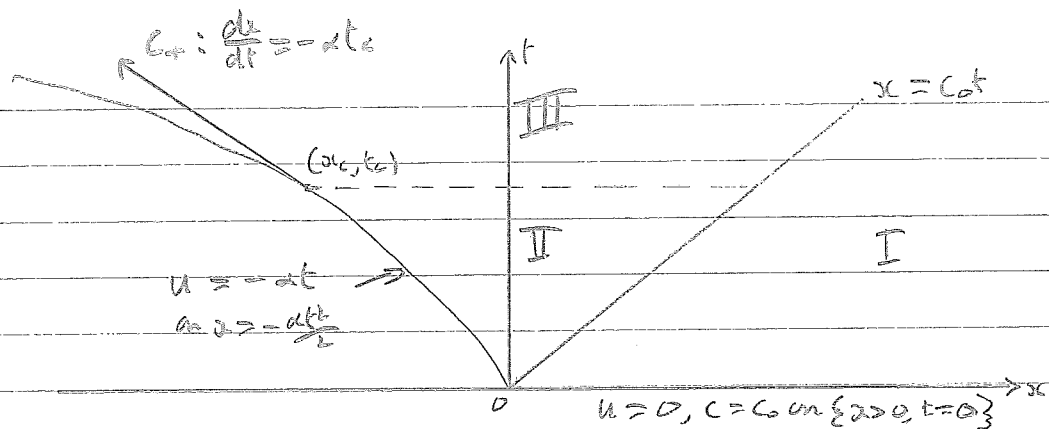
[B6]

are constant along c_\pm characteristic curves with $\frac{dx}{dt} = u \pm c$.

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\square

(b)



(i) Where C_{\pm} char \leq from $\{x > 0, t = 0\}$ intersect,

$$u \pm \frac{2c}{\gamma-1} = 0 \pm \frac{2c_0}{\gamma-1} \Rightarrow u = 0, c = c_0$$

[B/S2]

Hence such C_{\pm} char \leq are straight lines with $\frac{dx}{dt} = \pm c_0$ and therefore map out the region (I) $x > c_0 t$ for $t > 0$.

(ii) On a C_{+} char \leq originating from $(\alpha, t) = (-\frac{1}{2}\alpha Z^2, Z)$ on the piston, $u + \frac{2c}{\gamma-1} = R_{+}(Z) = \text{constant}$.

Where this C_{+} char \leq intersects the family of C_{-} char \leq from $\{x > 0, t = 0\}$,

$$u - \frac{2c}{\gamma-1} = 0 - \frac{2c_0}{\gamma-1}$$

so that $u = -\alpha Z$ (by BC on piston) and $c = c_0 + \frac{1}{2}(\gamma-1)(-\alpha Z)$ on this C_{+} char \leq , which is therefore a straight line with slope $\frac{dx}{dt} = u + c$, so given by

[S3]

$$x = -\frac{1}{2}\alpha Z^2 + \left(c_0 + \frac{1}{2}(\gamma-1)(-\alpha Z)\right)(t - Z)$$

This gives the solution parametrically. Eliminating $Z = -\frac{u}{\alpha}$

$$\Rightarrow x = -\frac{1}{2}\alpha\left(-\frac{u}{\alpha}\right)^2 + \left(c_0 + \frac{1}{2}(\gamma-1)u\right)\left(t + \frac{u}{\alpha}\right)$$

$$\Rightarrow \gamma u^2 + \left((\gamma+1)\alpha t + 2c_0\right)u + 2\alpha(c_0 t - x) = 0$$

$$\Rightarrow 2\alpha x = -\left((\gamma+1)\alpha t + 2c_0\right) \pm \left(\left((\gamma+1)\alpha t + 2c_0\right)^2 - 8\gamma\alpha(c_0 t - x)\right)^{\frac{1}{2}}$$

D

Since $\mathcal{I} = 0 \Rightarrow u = 0, x = ct$, we must choose the -root.

With $t_c = \frac{2c_0}{(\gamma-1)\alpha}$, the discriminant is given by

$$\begin{aligned} D &= (\gamma+1)^2 \alpha^2 t^2 + 4(\gamma+1)\alpha ct - 4c_0^2 - 8\gamma\alpha ct + 8\gamma\alpha a + 4\gamma\alpha^2 t^2 - 4\gamma t^2 \\ &= (\gamma-1)^2 \alpha^2 t^2 - 4(\gamma-1)\alpha ct + 4c_0^2 + 4\gamma\alpha(2a + \alpha t^2) \\ &= (\gamma-1)^2 \alpha^2 (t - t_c)^2 + 4\gamma\alpha(2a + \alpha t^2) \geq 0 \text{ for } a \geq -\frac{\alpha t^2}{2} \end{aligned}$$

Moreover, $u = \frac{2(c-c_0)}{\gamma+1}$, so

$$2\gamma u = \left((\gamma-1)^2 \alpha^2 (t - t_c)^2 + 4\gamma\alpha(2a + \alpha t^2) \right)^{\frac{1}{2}} - (\gamma+1)\alpha t + 2c_0 = \frac{4\gamma\alpha(c-c_0)}{\gamma-1} \quad (*)$$

[S/N5] gives solution in $-\frac{1}{2}\alpha t^2 \leq a \leq ct$ and it is physical so long as $c \propto \rho^{(\gamma-1)/2}$ is non-negative (in region II).

$(*) \Rightarrow u$ and c minimal on piston at $a = -\frac{1}{2}\alpha t^2$, where for $t < t_c$,

$$\begin{aligned} c &= c_0 + \frac{\gamma-1}{4\gamma} \left((\gamma-1)\alpha(t_c - t) - (\gamma+1)\alpha t - 2c_0 \right) \\ &= c_0 - \frac{\gamma-1}{4\gamma} \cdot 2\gamma\alpha t = \frac{\gamma-1}{2} \alpha (t_c - t), \end{aligned}$$

so that the soln $(*)$ is physical for $t < t_c$, but gas density vanishes on piston at time $t = t_c$. \square

[S/N2]

(iii) For $t > t_c$ the piston leaves the gas behind forming a vacuum between the piston and the last C_+ char s to leave the piston at $(a, t) = (-\frac{1}{2}\alpha t_c^2, t_c)$. On this char s , $u = \frac{2c}{\gamma-1} = -\alpha t_c$ since $u = -\alpha t_c, c = 0$ on piston at $(-\frac{1}{2}\alpha t_c^2, t_c)$, while $u = \frac{2c}{\gamma-1} = -\frac{2c_0}{\gamma-1}$ from C_- char s from $\{a \geq 0, t=0\} \Rightarrow u = -\alpha t_c, c = 0$ on C_+ char s $a = -\frac{1}{2}\alpha t_c^2 + (\gamma-1)\alpha t_c(t - t_c)$.

The analysis in part (ii) still holds for C_+ char s leaving piston before $t = t_c$, so the soln is again given by $(*)$ but now in region $-\frac{1}{2}\alpha t_c^2 - \alpha t_c(t - t_c) \leq a < ct$ for $t > t_c$ (region III in diagram). \square

[N3]

[5]