C7.5: General Relativity I
Problem Sheet 1

Please submit solutions in the collection boxes in the basement of the Mathematical Institute no later than 15th October, 1pm.

1. Practice with tensors

Given a tensor X^{ab} and a vector V^a with components

$$X^{ab} = \begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix}, \qquad V^a = (-1, 2, 0, -2),$$

find $X^a{}_b$, $X_a{}^b$, $X^{(ab)}$, $X_{[ab]}$, $X^a{}_a$, $V^a V_a$, and $V_a X^{ab}$, using $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$.

2. Summation convention

Explain why each equation below is ambiguous or inconsistent. Provide a possible correct version for each.

1. $x'^a = L^{ab} x^b$
2. $x'^a = L^b{}_c M^c{}_d x^d$
3. $\delta^a_b = \delta^a_c \delta^c_d$
4. $x'^a = L^a{}_c x^c + M^c{}_d x^d$
5. $x'^a = L^a{}_c x^c + M^{ad} x^d$
6. $\phi = (X^a A_a)(Y^a B_a)$

3. Operations on tensors

Consider two general coordinate systems $\{x^a\}$ and $\{x'^a\}$ on overlapping regions of flat Minkowski spacetime. How do the components of a (p, q) tensor transform under the change of coordinates from $\{x^a\} \mapsto \{x'^a\}$?

What form does the Jacobian matrix $\partial x'^a / \partial x^b$ take for a Lorentz transformation between two inertial frames?

- Show that the sum $S^a{}_b + T^a{}_b$ of two $(1,1)$ tensors is also a $(1,1)$ tensor.
- Show that the tensor product $S^a{}_b T^c$ of a $(1,1)$ tensor and an $(1,0)$ tensor is a $(2,1)$ tensor.
- Show that the contraction $S^{ac}{}_{bc}$ of a $(2,2)$ tensor is a $(1,1)$ tensor.
- Show that the partial derivative $\partial_a S^b$ of a $(1,0)$ tensor transforms as a $(1,1)$ tensor under Lorentz transformations between inertial frames, but not under general coordinate transformations.

4. Levi-Civita tensor

For any tensor $T^{a_1 \dots a_q}$, we may define its symmetrisation

$$T^{(a_1 \dots a_q)} \equiv \sum_{\sigma \in \mathcal{S}_q} T^{\, a_{\sigma(1)} \cdots a_{\sigma(q)}},$$

and anti-symmetrisation

$$T^{[a_1 \dots a_q]} \equiv \sum_{\sigma \in \mathcal{S}_q} \mathrm{sig}(\sigma)\, T^{\, a_{\sigma(1)} \cdots a_{\sigma(q)}},$$

where $\mathcal{S}_q$ is the group of permutations $\{\sigma\}$ of q elements and $\mathrm{sig}(\sigma)$ denotes the sign of a permutation σ.

Prove that a completely anti-symmetric $(0,m)$ tensor in n dimensions vanishes unless $m \leq n$. How many independent components does such a tensor have?

In a four-dimensional spacetime with metric g_{ab}, the Levi-Civita tensor ϵ_{abcd} is defined by two properties:

1. It is completely anti-symmetric: $\epsilon_{abcd} = \epsilon_{[abcd]}$.
2. $\epsilon_{0123} = \sqrt{-g}$ in a right-handed coordinate system $\{x^0, x^1, x^2, x^3\}$, where g is the determinant of the metric.

Show that $\epsilon_{0123} = 1$ in a right-handed inertial frame. Prove that ϵ_{abcd} transforms as a $(0,4)$ tensor under general coordinate transformations

$$x \to x'(x).$$

5. Maxwell's equations in an inertial frame

Show that if $F_{ab} = -F_{ba}$,

$$\partial_{[a} F_{bc]} = 0 \qquad \Leftrightarrow \qquad \partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0.$$

The electromagnetic field is encoded in an anti-symmetric $(0,2)$ tensor field, F_{ab}. The electric and magnetic fields measured by an observer with 4-velocity V^a are extracted from F_{ab} by

$$E_a = F_{ab} V^b, \qquad B_a = -\tfrac{1}{2} \epsilon_{abcd} F^{bc} V^d,$$

where ϵ_{abcd} is the Levi-Civita tensor. By contracting with the 4-velocity V^a, explain why E_a and B_a each have only 3 independent components. For an observer at rest in an inertial frame, so that $V^a = (1,0,0,0)$, show that

$$\begin{aligned} E_a &= (0, \vec{E}) \qquad \text{where } E_i = F_{i0}, \\ B_a &= (0, \vec{B}) \qquad \text{where } B_i = \frac{1}{2}\epsilon_{ijk}F^{jk}. \end{aligned}$$

Hence show that

$$\partial_a F^{ab} = -4\pi J^b, \qquad \partial_{[a}F_{bc]} = 0,$$

reproduce Maxwell's equations for the electromagnetic fields $(\vec{E}, \vec{B})$. The vector field J^a has components $(\rho, \vec{J})$, where ρ is the electric charge density and $\vec{J}$ is the electric current density measured by an observer at rest in the inertial frame. Show that $\partial_a J^a = 0$ and explain the physical meaning of this equation.

The contribution to the energy-momentum tensor from the electromagnetic field is

$$T^{ab} = \frac{1}{4\pi}\Big[F^{ac}F^b{}_c - \tfrac{1}{4}(F^{cd}F_{cd})\eta^{ab}\Big].$$

Assuming $J^a = 0$, show that this energy-momentum tensor is conserved, $\partial_a T^{ab} = 0$. What happens when $J^a \neq 0$?

6. Geodesics and motion in an EM field

Consider a curve $x^a(\lambda)$ in flat Minkowski space parametrised by a real parameter $\lambda_1 \leq \lambda \leq \lambda_2$. What is the condition for this curve to be time-like?

The proper time of a time-like curve is the time measured by an observer moving along it. In an inertial reference frame, this is given by the functional

$$\Delta\tau = \int_{\lambda_1}^{\lambda_2} \mathrm{d}\lambda \sqrt{-\eta_{ab}\,\frac{\mathrm{d}x^a}{\mathrm{d}\lambda}\frac{\mathrm{d}x^b}{\mathrm{d}\lambda}},$$

where $\eta_{ab} = \mathrm{diag}(-1,1,1,1)$. Consider the variational problem for this functional and show that the curve of extremal proper time between two points $x^a(\lambda_1)$ and $x^a(\lambda_2)$ is a straight line. Is this a minimum or a maximum?

Show that one may always reparametrise the curve such that $\sqrt{-\eta_{ab}\,\dot{x}^a\dot{x}^b}$ is constant, where $\dot{x}^a = \mathrm{d}x^a/\mathrm{d}\lambda$. What is the parametrisation that achieves this? Such a parameter is called an *affine* parameter.

Why is extremising the functional

$$S = -\frac{1}{2}\int_{\lambda_1}^{\lambda_2} \mathrm{d}\lambda\, \eta_{ab}\,\frac{\mathrm{d}x^a}{\mathrm{d}\lambda}\frac{\mathrm{d}x^b}{\mathrm{d}\lambda},$$

equivalent to extremising the proper time when using an affine parameter? What is left of the reparametrisation freedom $\lambda \to \lambda'(\lambda)$ when working with this action, that is what are the relations between choices of affine parameters?

Now consider the modified functional

$$S = -\int_{\lambda_1}^{\lambda_2} \mathrm{d}\lambda \left[\frac{m}{2}\,\eta_{ab}\,\frac{\mathrm{d}x^a}{\mathrm{d}\lambda}\frac{\mathrm{d}x^b}{\mathrm{d}\lambda} - qA_a\frac{\mathrm{d}x^a}{\mathrm{d}\lambda} \right].$$

Show that the solution of the variational problem is

$$\frac{\mathrm{d}^2x^a}{\mathrm{d}\lambda^2} = \frac{q}{m}F^a{}_b\frac{\mathrm{d}x^b}{\mathrm{d}\lambda} \qquad \text{where } F_{ab} = \partial_a A_b - \partial_b A_a.$$

This equation describes the motion of a particle of mass m and electric charge q in an electromagnetic field F_{ab}. Contract the equation of motion with $\dot{x}^a$ and show that $\sqrt{-\eta_{ab}\,\dot{x}^a\dot{x}^b}$ remains constant in the presence of an electromagnetic field.