8.1 - Riemann tensor

Recall $[\nabla_a, \nabla_b] V^d = R_{abc}{}^d V^c$

Extends to other tensors by induction

1) $[\nabla_a, \nabla_b]\phi = [\partial_a, \partial_b]\phi - (\Gamma^c{}_{ab} - \Gamma^c{}_{ba})\partial_c\phi$

$= 0$

2) Apply to $\phi = V^a \omega_a$ to read off action

$$[\nabla_a, \nabla_b] T^{c_1 \dots}{}_{d_1 \dots} = R_{ab}{}^{c_1}{}_e T^{e c_2 \dots}{}_{d_1 \dots} + \dots$$

$$- R_{ab}{}^e{}_{d_1} T^{c_1 \dots}{}_{e d_2 \dots} + \dots$$

(Think of $R_{ab}{}^c{}_d$ as $(R_{ab})^c{}_d$, then acts with + on vector index and − on covectors)

Expression for $R_{ab}{}^c{}_d$ in terms of g_{ab}

$$[\nabla_a, \nabla_b] V^c = \partial_a \nabla_b V^c - \Gamma^d{}_{ab} \nabla_d V^c + \Gamma^c{}_{ad} \nabla_b V^d$$

$$- (a \leftrightarrow b)$$

$$= \partial_a(\partial_b V^c + \Gamma^c{}_{bd} V^d) - \Gamma^d{}_{ab}(\partial_d V^c + \Gamma^c{}_{de} V^e)$$

$$+ \Gamma^c{}_{ad}(\partial_b V^d + \Gamma^d{}_{be} V^e) - (a \leftrightarrow b)$$

Simplify using $[\partial_a, \partial_b] = 0$ and $\Gamma^a{}_{[bc]} = 0$

- All derivatives of V drop out

$$= \left(\partial_a \Gamma^c{}_{be} + \Gamma^c{}_{ad} \Gamma^d{}_{be} - a \leftrightarrow b \right) V^e$$

$$\equiv R_{ab}{}^c{}_e V^e$$

Comments : 1) Explicitly linear in $V^e \Rightarrow$ tensorial (verifying using transformation law is painful).
2) Depends on Γ and $\partial \Gamma$

3) Can pick $\Gamma|_p = 0$ but $\partial \Gamma|_p \neq 0$, so R does not vanish in local inertial frame in general.
4) $R = 0$ in cartesian for Minkowski, so also zero in any coordinates - converse also holds!

8.2 - Algebraic identities

1) $R_{ab}{}^c{}_d = - R_{ba}{}^c{}_d$ (from definition)

2) $R_{abcd} = - R_{abdc}$ (∇_a metric compatible)

Since $\nabla_a g_{bc} = 0$

$$0 = [\nabla_a, \nabla_b] g_{cd}$$

$$= R_{abcd} + R_{abdc}$$

3) $R_{[abc]d} = 0$ "First Bianchi identity"

(Torsion free $\Gamma^a{}_{[bc]} = 0$)

Compute $\nabla_{[a} \nabla_b \nabla_{c]} \phi = 0$ (uses torsion free)

But equal to $R_{[ab}{}^d{}_{c]} \nabla_d \phi$

As this vanishes for all ϕ, $R_{[ab}{}^d{}_{c]} = 0$

$\Rightarrow R_{[abc]d} = 0$

1) + 2) + 3) $\Rightarrow$ $R_{abcd} = R_{cdab}$

8.3 – Bianchi identity

$\nabla_{[a} R_{bc]}{}^d{}_e = 0$ (like $\nabla_{[a} F_{bc]} = 0$)

(needs torsion free)

1) $(\nabla_a \nabla_b - \nabla_b \nabla_a) \nabla_c \omega_d$

$= - R_{ab}{}^e{}_c \nabla_e \omega_d - R_{ab}{}^e{}_d \nabla_c \omega_e$

2) $\nabla_c (\nabla_a \nabla_b - \nabla_a \nabla_b) \omega_d$

$= \nabla_c (-R_{ab}{}^e{}_d \omega_e)$

$= - \nabla_c R_{ab}{}^e{}_d \omega_e - R_{ab}{}^e{}_d \nabla_c \omega_e$

Antisymmetrise 1) and 2) over a, b, c

- LHS are equal so

vanishes due to 1st Bianchi

$$- R_{[ab}{}^{e}{}_{c]} \nabla_e \omega_d - R_{[ab|}{}^{e}{}_{d} \nabla_{|c]} \omega_e$$

cancel

$$= - \nabla_{[c} R_{ab]}{}^{e}{}_{d} \, \omega_e - R_{[ab|}{}^{e}{}_{d} \nabla_{|c]} \omega_e$$

must vanish for all ω_e

8.4 - Einstein equation

So far we have described the effects of a gravitational field on a test particle

- Like $\ddot{\underline{x}} = - \underline{\nabla} \underline{\Phi}$

Gravity is non-linear, the gravitational field should satisfy a generalisation of the Poisson equation, $\nabla^2 \Phi \sim 4\pi G \rho$

That is, for a given distribution of matter and energy, what is the resulting metric?

Should be of the form

$$\text{Tensor}(g_{ab}) = \text{Tensor}(\text{matter, energy} \ldots)$$

Stress tensor T^{ab} captures source contribution

$$\Rightarrow \quad G_{ab} = \lambda T_{ab}$$

where
- G_{ab} linear in $R_{ab}{}^{c}{}_{d}$ (2^{nd} order equations of motion for the metric)
- $G_{ab} = G_{ba}$ (as T symmetric)
- $\nabla^a G_{ab} = 0$ (as $\nabla^a T_{ab} = 0$)

($G_{ab} \propto g_{ab}$ doesn't work as $\operatorname{tr} g = 2$ but $\operatorname{tr} T_{ab}$ can vanish)

There is (almost) a unique answer to this question!

Define: $R_{ab} = R_{acb}{}^{c} = R_{acbd}\, g^{cd}$

"Ricci tensor" ($R_{ab} = R_{ba}$)

$R = g^{ab} R_{ab}$

"Ricci scalar"

Now start from Bianchi identity and contract

$\nabla_{[a} R_{bc]de} = 0$

$\equiv \quad \nabla_a R_{bcde} + \nabla_b R_{cade} + \nabla_c R_{abde} = 0$

$\times\ g^{bd} g^{ce}$ and use $\nabla g = 0$

$= \quad \nabla_a R_{bc}{}^{bc} + \nabla_b R_{ca}{}^{bc} + \nabla_c R_{ab}{}^{bc}$

$= \quad \nabla_a R \qquad - \nabla_b R_a{}^b - \nabla_c R_a{}^c$

$= \quad 0$

$$\Rightarrow \quad \nabla^a (g_{ab} R - 2 R_{ab}) = 0$$

i.e.

$$\nabla^a \underbrace{\left(R_{ab} - \tfrac{1}{2} g_{ab} R \right)}_{G_{ab}} = 0$$

G_{ab} known as <u>Einstein tensor</u>.

$$R_{ab} - \tfrac{1}{2} g_{ab} = \lambda T_{ab}$$

<u>"Einstein equation"</u>

Fix λ using Newtonian limit.

Example : S^2 with round metric

Recall $R_{\theta\phi}{}^{\phi}{}_{\theta} = -1$

$$g^{\theta\theta} = 1 \qquad g_{\theta\theta} = 1$$

$$g^{\phi\phi} = \frac{1}{\sin^2\theta} \qquad g_{\phi\phi} = \sin^2\theta$$

$$R_{\theta\theta} = R_{\theta c \theta}{}^{c}$$

$$= R_{\theta\phi\theta}{}^{\phi}$$

$$= +1$$

$$R_{\phi\phi} = R_{\phi\theta\phi}{}^{\theta}$$

$$= R_{\phi\theta}{}^{\phi}{}_{\theta} \, g_{\phi\phi} \, g^{\theta\theta}$$

$$= (+1) \sin^2\theta$$

i.e. $R_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$

Note $R_{ab} = g_{ab}$! "Einstein metric"

Scalar $R = g^{ab} R_{ab} = g^{\theta\theta} R_{\theta\theta} + g^{\phi\phi} R_{\phi\phi}$

$$= 2$$

Einstein tensor : $G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R$

$= 0$ identically.

True for any metric in two dimensions.

Comments : 1) $R \sim \partial \Gamma + \Gamma^2$

$\sim \partial^2 g + \partial g \, \partial g$

R is 2^{nd} order in derivatives, as is G_{ab}.

Thus the equation for g_{ab} is 2^{nd} and g_{ab} is a dynamical field.

2) One can derive Einstein equations from a 4d action

$$S = \int d^4x \sqrt{-g} \, R$$

$\delta S = 0$ reproduces $G_{ab} = 0$

Can add "matter fields to generate T_{ab}.