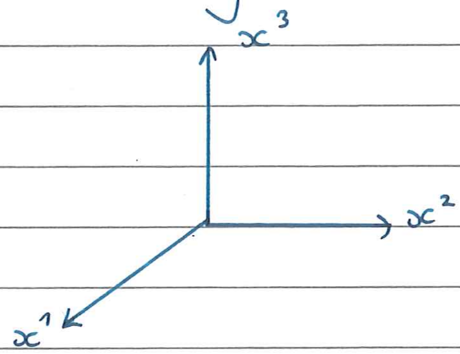
General Relativity: Lecture 1

1.1 - Newtonian theory

Physical laws on $\mathbb{R}^3 \times \mathbb{R}^1$ with universal time t

Inertial frame: coordinates $\{x^1, x^2, x^3\}$ for non-accelerating observer.



Transformation between inertial frames

$$x^i \mapsto x'^i = R^i{}_j x^j - v^i t + a^i$$

$R^i{}_j$: rotation SO(3)
v^i : relative velocity
a^i : spatial translation

$$t \mapsto t' = t + b$$

b : time translation

Principle of Relativity: laws of physics independent of choice of inertial frame.

Gravity :
- gravitational field affects motion of particles
- field generated by a mass distribution.

Gravitational potential

$$\Phi : \mathbb{R}^3 \to \mathbb{R}$$
$$x^i \mapsto \Phi(x)$$

obeys Poisson equation and produces acceleration

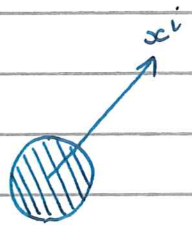
$$\nabla^2 \Phi = 4\pi G \rho \,, \quad m \frac{d^2}{dt^2} \underline{x} = m \ddot{\underline{x}} = -m \underline{\nabla} \Phi$$

where $\rho : \mathbb{R}^3 \to \mathbb{R}$ is matter density

G : Newton's constant, "strength of gravity"

∇^2 : Laplacian on $\mathbb{R}^3$, $\delta^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j}$

Example : localised spherical mass



$$M = \int_0^R d^3x \, \rho$$

$$\Phi = -\frac{GM}{|\underline{x}|}$$

$$\Rightarrow \ddot{x}^i = -\frac{GM}{|\underline{x}|^2} \frac{\underline{x}}{|\underline{x}|}$$

1.2 - Problems with Newtonian theory

a) Instantaneous signal propagation

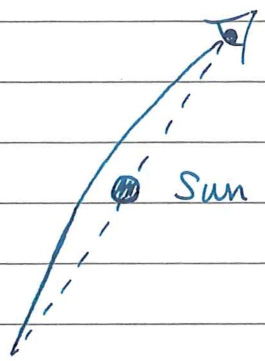
No time derivatives in Poisson equation
⇒ instant signalling

Incompatible with Special Relativity (signals limited by c)

b) Bending of light

If light is a wave, no coupling between electromagnetism and gravity, so no deflection.

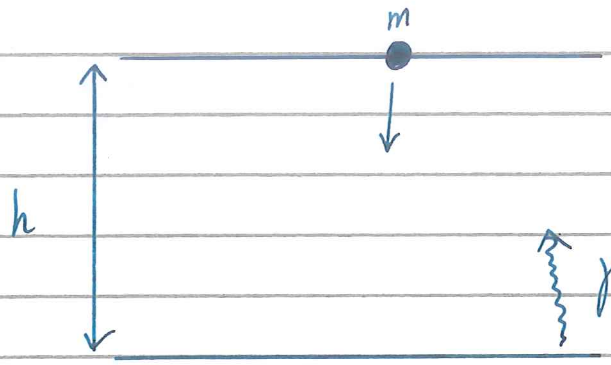
If light is a particle (massless test particle), get 1/2 of deflection observed



Eddington 1919

Wait for solar eclipse then compare location of stars near sun.

c) Thought experiment (Einstein)



- Drop mass m from height h
- A bottom, convert m to energy as photon
- Photon measured again at top.

$$E_m^{top} = mc^2$$

$$E_m^{bottom} = mc^2 + \frac{1}{2}mv^2 + O(v^4)$$

$$= mc^2\left(1 + \frac{gh}{c^2} + \dots\right)$$

↑ SR corrections.

$g \simeq 10\ ms^{-2}$, grav. acc. at Earth's surface.

$\frac{gh}{c^2} \ll 1$ for small SR effects. (linear terms)

$$E_m^{bottom} = E_\gamma^{bottom} \simeq mc^2(1 + gh/c^2)$$

Conservation of energy $\Rightarrow E_\gamma^{top} = mc^2$

But $\frac{E_\gamma^{top}}{E_\gamma^{bottom}} \simeq \frac{1}{1 + gh/c^2} \simeq 1 - \frac{gh}{c^2}$

So energy (and frequency) of photon must decrease on way back up - "<u>gravitational redshift</u>".

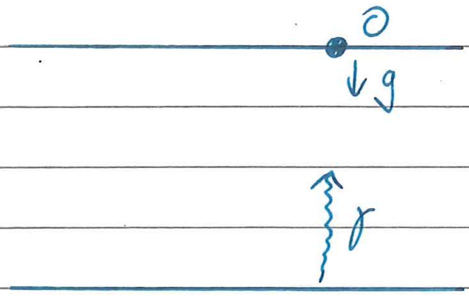
Clock at bottom runs slow! "Time dilation"

Frame at rest is <u>not</u> inertial

No mechanism for this to happen in Newtonian gravity!

- gravity couples to <u>mass</u>, not energy.
- energy conserved, earth ensures momentum conserved.
- redshift observed by Pound - Rebka 1960

Consider an observer O falling freely from to



$t = 0$: photon released from bottom, O at rest

$t = h/c$: photon reaches top, O has speed $u = gh/c$

There will be a SR velocity redshift for O

$$E_\gamma^{O} = \left(\frac{1 + u/c}{1 - u/c}\right)^{1/2} E_\gamma^{top} \simeq (1 + u/c)\, E_\gamma^{top}$$

$$\simeq \left(1 + \frac{gh}{c^2}\right) E_\gamma^{top}$$

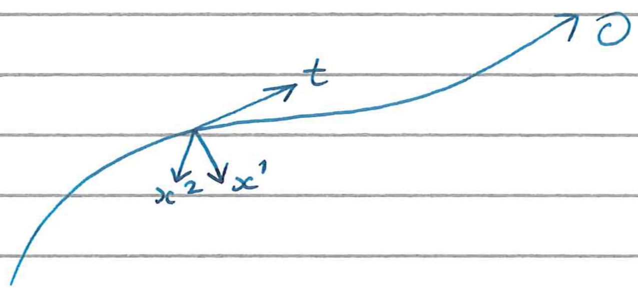
So $$\frac{E_\gamma^{O}}{E_\gamma^{bottom}} = \frac{E_\gamma^{O}}{E_\gamma^{t}} \frac{E_\gamma^{t}}{E_\gamma^{b}} \simeq \frac{1 + gh/c^2}{1 + gh/c^2} \simeq 1 + O(g^2h^2)$$

The redchift of the photon <u>vaniches</u> in the freely falling reference frame.

<u>Equivalence Principle</u>: may eliminate effects of gravity locally by moving to a freely falling reference frame

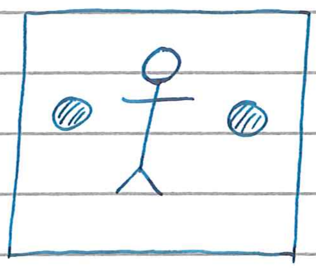
i.e. local affects of gravity indistinguishable from bring in an accelerated frame of reference.

A freely falling observer $\mathcal{O}$ may use a <u>local</u> inertial frame $\{x^\mu\}$ in a small neighbourhood of a point on their worldline

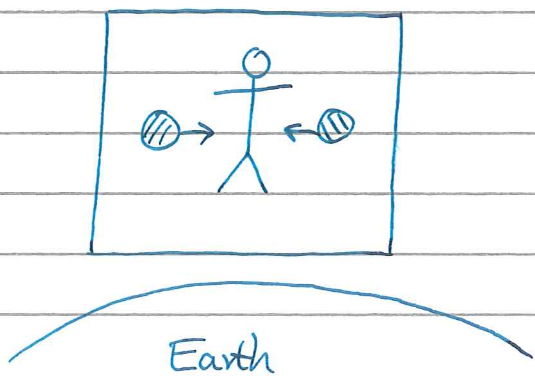


Special Relativity holds in this local frame.

<u>Example</u>: elevator falling to earth



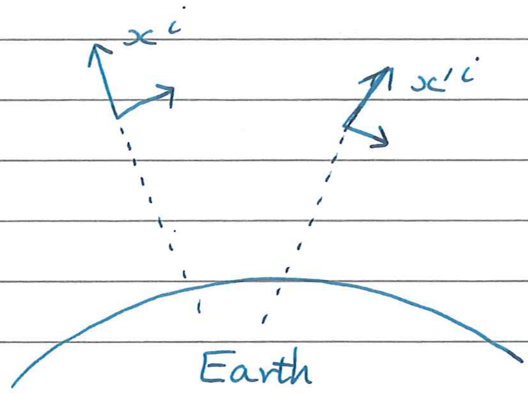
"constant field"



If elevator is small and you only consider a short time, these look the same!

<u>Local</u>: small neighbourhood in space and time.

Gravity appears as <u>relative acceleration</u> of local inertial frames



Freely falling frames approach each other

Transformation between these frames is not a (linear) Lorentz transformation

$$x'^a \neq L^a{}_b x'^b$$

$\Rightarrow$ Spacetime is <u>curved</u>.

Need a new theory :- SR holds over short distances / times

- Equations will be non-linear (energy sources field)

- Reduces to Newtonian gravity for weak fields and low speeds.