C5.12 2017 q3

3/ (a) $C \underset{\beta}{\overset{\alpha}{\rightleftharpoons}} O$ ← fraction n

$\dot{n} = \alpha(1-n) - \beta n$

$\Rightarrow \tau \dot{n} = n_\infty - n \qquad \tau = \frac{1}{\alpha+\beta}, \quad n_\infty = \frac{\alpha}{\alpha+\beta}$

τ_n relaxation time

n_∞ equilibrium

(b) $S_0 \underset{\beta}{\overset{l\alpha}{\rightleftharpoons}} S_1 \underset{2\beta}{\overset{(l-1)\alpha}{\rightleftharpoons}} \cdots \quad S_{l-1} \underset{l\beta}{\overset{\alpha}{\rightleftharpoons}} S_l$

(i) The reaction is $S_j \underset{(j+1)\beta}{\overset{(l-j)\alpha}{\rightleftharpoons}} S_{j+1}$

because S_j has j open gates, $l-j$ closed so $(l-j)$ chances of opening

S_{j+1} has $j+1$ so ~~has~~ $j+1$ chances of closing

(ii) $\cdots S_{j-1} \underset{j\beta}{\overset{(l-j+1)\alpha}{\rightleftharpoons}} S_j \underset{(j+1)\beta}{\overset{(l-j)\alpha}{\rightleftharpoons}} S_{j+1} \cdots$

R_j R_{j+1}

x_j = fraction of ~~open~~ S_j channels

write $R_j = (l-j+1)\alpha x_{j-1} - j\beta x_j \qquad j = 1, \ldots l$

then $\dot{x}_j = R_j - R_{j+1} \qquad j = 1, \ldots l-1$

Also

$$\dot{x}_0 = -R_1$$
$$\dot{x}_\ell = R_\ell$$

(iii) To show a solution is $x_j = \binom{\ell}{j} n^j (1-n)^{\ell-j}$

a cute way is then

$$\cancel{(1+x_j s)} \qquad \sum_0^\ell x_j s^j = \sum_0^\ell \binom{\ell}{j} (ns)^j (1-n)^{\ell-j}$$
$$= (1-n+ns)^\ell$$

So define $\underline{\phi(s,t) = \sum_0^\ell x_j s^j}$

$$\text{then } \phi_t = \sum_0^\ell \dot{x}_j s^j$$
$$= \sum_0^\ell \left[R_j - R_{j+1} \right] s^j \qquad \text{providing we define } R_0 = R_{\ell+1} = 0$$
$$= \sum_0^\ell R_j s^j - \frac{1}{s} \sum_0^\ell R_{j+1} s^{j+1}$$
$$= \sum_1^\ell R_j s^j - \frac{1}{s} \sum_0^{\ell-1} R_{j+1} s^{j+1}$$
$$= \sum_1^\ell R_j s^j - \frac{1}{s} \sum_1^\ell R_k s^k$$
$$= \left(1 - \frac{1}{s}\right) \sum_1^\ell R_j s^j$$

$$\text{Also } \sum_1^\ell R_j s^j = \sum_1^\ell (\ell - j + 1) \alpha x_{j-1} s^j - \sum_1^\ell \beta j x_j s^j$$
$$= \sum_0^{\ell-1} (\ell - k) \alpha x_k s^{k+1} - \beta s \sum_1^\ell j x_j s^{j-1}$$

$(\ell - \ell = 0)$: $$= \sum_0^\ell (\ell - k) \alpha x_k s^{k+1} - \beta s \sum_0^\ell j x_j s^{j-1}$$ $(0 = 0)$

$$= \sum_0^\ell \alpha \ell s \, x_k s^k - \sum_0^\ell \alpha s^2 k x_k s^{k-1} - \beta s \sum_0^\ell j x_j s^{j-1}$$
$$= \alpha \ell s \phi - \alpha s^2 \phi_s - \beta s \phi_s$$

Hence

$$\phi_t = \left(1-\frac{1}{s}\right)\left[\alpha l s\phi - (\alpha s^2+\beta s)\phi_s\right]$$

$$= (s-1)\left[\alpha l \phi - (\alpha s+\beta)\phi_s\right]$$

& if we put $\phi = (1-n+ns)^l$

$$\phi_t = l(1-n+ns)^{l-1}(s-1)\dot{n}$$

$$\phi_s = l(1-n+ns)^{l-1}\, n$$

& this solves the pde (& thus the x_j eqs) if

$$\phi_t = l(1-n+ns)^{l-1}(s-1)\dot{n}$$

$$= (s-1)\left[\alpha l\phi - (\alpha s+\beta)\phi_s\right]$$

$$= (s-1)\left[\alpha l(1-n+ns)^l - (\alpha s+\beta)\, l(1-n+ns)^{l-1} n\right]$$

i.e. [$\div$ by $l(1-n+ns)^{l-1}(s-1)$]

if $\quad \dot{n} = \alpha(1-n+ns) - (\alpha s+\beta)n$

$$= \alpha(1-n) - \beta n$$

(c)

$$\begin{tikzcd} g \arrow[d] & & n \arrow[d] \\ C \arrow[rr, shift left, "\alpha"] \arrow[dr, "\delta"'] & & O \arrow[ll, shift left, "\beta"] \arrow[dl, "\gamma"] \\ & I & \end{tikzcd}$$

(i) $1-n-g$

(ii) $\dot{n} = \alpha g - \beta n - \gamma n$

$\dot{g} = -\alpha g + \beta n - \delta g$

(iii) $t=0$ $n=1$ $g=0$ all open

$$\begin{pmatrix} \dot{n} \\ \dot{g} \end{pmatrix} = \begin{pmatrix} -(\beta+\gamma) & \alpha \\ \beta & -(\alpha+\delta) \end{pmatrix} \begin{pmatrix} n \\ g \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} n \\ g \end{pmatrix} \sim \underset{\sim}{c_1} e^{\lambda_1 t} + \underset{\sim}{c_2} e^{\lambda_2 t}$$

$$(\lambda + \beta + \gamma)(\lambda + \alpha + \delta) - \alpha\beta = 0 \Rightarrow \lambda_1, \lambda_2$$

$\Rightarrow g = a(e^{\lambda_1 t} - e^{\lambda_2 t})$ $(\lambda_1 > \lambda_2)$ for $g=0$ at $t=0$

~~[illegible]~~

then (g eq) $n = \frac{1}{\beta}[\dot{g} + (\alpha+\delta)g]$

so at $t=0$ $n = 1 = \frac{1}{\beta}[\dot{g}|_0] = \frac{a}{\beta}(\lambda_1 - \lambda_2)$

so $a = \dfrac{\beta}{\lambda_1 - \lambda_2}$

[(note $\lambda^2 - \underbrace{(\alpha+\beta+\gamma+\delta)}_{T}\lambda + \underbrace{\alpha\gamma + \beta\delta + \delta\gamma}_{D} = 0$

so ~~$\lambda_1+\lambda_2 = \alpha+\beta+\gamma+\delta$~~ $\lambda_{1,2} = \frac{1}{2}[T \pm (T^2 - 4D)^{1/2}]$

$\Rightarrow \lambda_1 - \lambda_2 = (T^2 - 4D)^{1/2} = [(\alpha+\beta+\gamma+\delta)^2 - 4(\alpha\gamma+\beta\delta+\gamma\delta)]^{1/2}$]