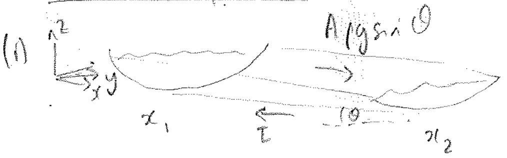
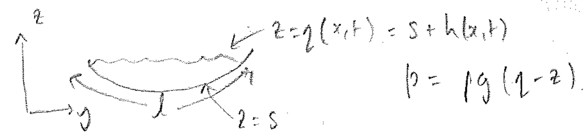
Sheet 3

1. St Venant equations

(i) 



$p = \rho g(\eta - z)$

Mass conservation $\displaystyle \frac{d}{dt}\int_{x_1}^{x_2} A\,dx = Q|_{x_1} - Q|_{x_2} = -\int_{x_1}^{x_2}\frac{\partial Q}{\partial x}\,dx$ where $\boxed{Q = Au}$ is volume flux.

(water is incompressible so volume is conserved).

Since x_1, x_2 are arbitrary, this implies $\boxed{\dfrac{\partial A}{\partial t} + \dfrac{\partial Q}{\partial x} = 0.}$ ①

(& assuming continuity)

Momentum conservation
$$\frac{d}{dt}\int_{x_1}^{x_2} \rho A u\,dx = \underbrace{\rho A u^2|_{x_1} - \rho A u^2|_{x_2}}_{\text{momentum fluxes}} - \underbrace{\int_{x_1}^{x_2} \tau l}_{\text{friction}} + \underbrace{\rho g A \sin\theta\, dx}_{\text{weight}} + \underbrace{\bar{p}A|_{x_1} - \bar{p}A|_{x_2}}_{\text{pressure forces}},$$

$S = \sin\theta$
$$= \int_{x_1}^{x_2} -\frac{\partial}{\partial x}(Au^2) - \tau l + \rho g A S - \frac{\partial}{\partial x}(\bar{p}A)\,dx$$

Since x_1, x_2 are arbitrary, $\displaystyle \frac{\partial}{\partial t}(\rho A u) + \frac{\partial}{\partial x}(\rho A u^2) = \rho g A S - \tau l - \frac{\partial}{\partial x}(\bar{p}A)$

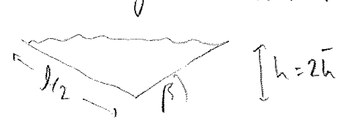
Since $p = \rho g(\eta - z)$, $\displaystyle \bar{p}A = \int_A \rho g(\eta - z)\,dy\,dz$ $\displaystyle \frac{\partial}{\partial x}(\bar{p}A) = \int_A \rho g \eta_x\,dy\,dz = \rho g A \eta_x$, assuming η_x is independent of y. Also assume s flat, so $\eta_x = \bar{h}_x$, $\bar{h}$ = average depth.

Using ①, dividing by ρA $\Rightarrow$ $\boxed{\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = gS - \dfrac{\tau l}{\rho A} - g\bar{h}_x}$ ②.

Steady uniform flow would have $gS = \dfrac{\tau l}{\rho A}$, $\Rightarrow$ $\tau = \rho g S R$ where $R = \dfrac{A}{l}$

If this has $u = \dfrac{R^{2/3}S^{1/2}}{n}$ then (eliminating S) $\boxed{\tau = \dfrac{\rho g n^2}{R^{1/3}}u^2}$

For a triangular cross-section, $h = \frac{l}{2}\sin\beta$, $A = \left(\frac{l}{2}\right)^2\sin\beta\cos\beta = \frac{1}{8}l^2\sin(2\beta)$. $\Rightarrow$ $l = A^{1/2}\left(\frac{\sin 2\beta}{8}\right)^{-1/2}$



so $R = \dfrac{A}{l} = \dfrac{1}{8}l\sin 2\beta = \left(\dfrac{\sin 2\beta}{8}\right)^{1/2}A^{1/2}$ and $\bar{h} = \dfrac{\sin\beta}{4}\left(\dfrac{\sin 2\beta}{8}\right)^{-1/2}A^{1/2}$.

Hence $\boxed{\tau = \rho g n^2\left(\dfrac{\sin 2\beta}{8}\right)^{-1/6}\dfrac{u^2}{A^{1/6}}}$ and $\boxed{\bar{h} = \left(\dfrac{\tan\beta}{4}\right)^{1/2}A^{1/2}}$ $\quad \dfrac{\tau l}{\rho A} = \dfrac{\tau}{\rho R} = \dfrac{gn^2}{R^{4/3}}u^2 = gn^2\left(\dfrac{\sin 2\beta}{8}\right)^{-2/3}\dfrac{u^2}{A^{2/3}}$

Substituting into ②, we have
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = gS - gn^2\left(\frac{\sin 2\beta}{8}\right)^{-2/3}\frac{u^2}{A^{2/3}} - g\left(\frac{\tan\beta}{4}\right)^{1/2}\frac{\partial}{\partial x}(A^{1/2})$$

(ii) Non-dimensionalise. $x = L\hat{x}$, $A = [A]\hat{A}$, $u = [u]\hat{u}$, $t = [t]\hat{t}$

$$gS = gn^2\left(\frac{\sin 2\beta}{8}\right)^{-2/3}\frac{[u]^2}{[A]^{2/3}} \quad \& \quad [A][u] = Q \rightarrow [u], [A]$$

$$[t] = \frac{L}{[u]} \quad \rightarrow [t]$$

Solving for $[u]$ $\Rightarrow$ $$SQ^{2/3} = n^2\left(\frac{\sin 2\beta}{8}\right)^{-2/3}[u]^{8/3} \Rightarrow [u] = S^{3/8}Q^{1/4}\left(\frac{\sin 2\beta}{8}\right)^{1/4} n^{-3/4}.$$

$$[A] = Q^{3/4}S^{-3/8}\left(\frac{\sin 2\beta}{8}\right)^{-1/4} n^{3/4}.$$

Then non-dimensional equations are
$$\boxed{\begin{aligned}&\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}(Au) = 0\\ &\varepsilon F^2\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = 1 - \frac{u^2}{A^{2/3}} - \varepsilon\frac{\partial}{\partial x}\left(A^{1/2}\right)\end{aligned}}$$

where $\boxed{\varepsilon = \left(\frac{\tan\beta}{4}\right)^{1/2}\frac{[A]^{1/2}}{LS}} = \frac{[h]}{LS}$ and $\boxed{F^2 = \frac{[u]^2}{g[h]}}$ $\left([h] = \left(\frac{\tan\beta}{4}\right)^{1/2}[A]^{1/2}\right)$

(iii) If $\varepsilon \ll 1$ and $F \ll 1$, to leading order $A^{2/3} = u^2 \Rightarrow u \approx A^{1/3}$.

Then to next order in ε, $$u^2 = A^{2/3} - \varepsilon A^{2/3}\frac{\partial}{\partial x}(A^{1/2}) = A^{2/3} - \frac{1}{2}\varepsilon A^{1/6}\frac{\partial A}{\partial x}$$

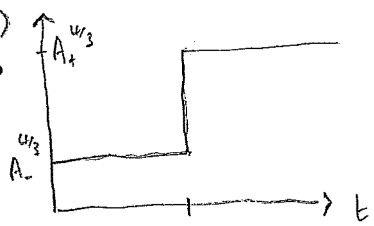
$$\Rightarrow u = A^{1/3}\left(1 - \frac{\varepsilon}{2}A^{-1/2}\frac{\partial A}{\partial x}\right)^{1/2}$$
$$\approx A^{1/3}\left(1 - \frac{\varepsilon}{4}A^{-1/2}\frac{\partial A}{\partial x} + \dots\right) \approx A^{1/3} - \frac{\varepsilon}{4}A^{-1/6}\frac{\partial A}{\partial x}.$$

So mass equation becomes $$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}\left[A^{4/3} - \frac{\varepsilon}{4}A^{5/6}\frac{\partial A}{\partial x}\right] = 0$$

or $$\boxed{\frac{\partial A}{\partial t} + \frac{4}{3}A^{1/3}\frac{\partial A}{\partial x} = \frac{\varepsilon}{4}\frac{\partial}{\partial x}\left(A^{5/6}\frac{\partial A}{\partial x}\right)}$$

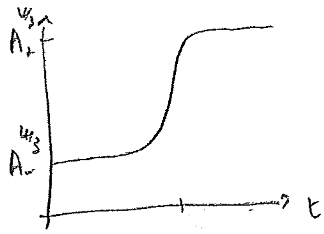
(iv) Q

(i) $\varepsilon = 0$



Shock.

(ii) $0 < \varepsilon \ll 1$



Smoothed-out shock.

2. Surface waves

(i) As in previous question, and choosing $L = [h]/S$ so as to make $\varepsilon = 1$, the equations for a triangular-shaped cross-section with Manning's law take the form

$$\boxed{\begin{aligned} &\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}(Au) = 0 \\ &F^2\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = 1 - \frac{u^2}{A^{2/3}} - \frac{1}{2A^{1/2}}\frac{\partial A}{\partial x} \end{aligned}} \qquad F^2 = \frac{[u]^2}{g[h]}.$$

Uniform steady state has $u^2 = A^{2/3}$ and if discharge is 1, $Au = 1$

$$\Rightarrow u = A = 1.$$

(ii) Perturbations $u = 1 + U$, $A = 1 + a$ with $U, a \ll 1$. Linearized equations

$$\boxed{\begin{aligned} &\frac{\partial a}{\partial t} + \frac{\partial a}{\partial x} + \frac{\partial U}{\partial x} = 0. \\ &F^2\left(\frac{\partial U}{\partial t} + \frac{\partial U}{\partial x}\right) = -2U + \frac{2}{3}a - \frac{1}{2}\frac{\partial a}{\partial x}. \end{aligned}} \rightarrow \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)a = -\frac{\partial U}{\partial x}$$

$$\Rightarrow F^2\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)^2 U = -2\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)U - \frac{2}{3}\frac{\partial U}{\partial x} + \frac{1}{2}\frac{\partial^2 U}{\partial x^2}$$

Solutions of form $U = e^{\frac{\sigma t + ikx}{F^2}}$ here $\quad \dfrac{F^2(\sigma + ik)^2}{F^4} = -\dfrac{2(\sigma + ik)}{F^2} - \dfrac{2}{3}\dfrac{ik}{F^2} - \dfrac{1}{2}\dfrac{k^2}{F^4}$

$$\Rightarrow (\sigma + ik)^2 + 2(\sigma + ik) + \frac{2}{3}ik + \frac{1}{2}\frac{k^2}{F^2} = 0$$

$$\sigma + ik = -1 \pm \underbrace{\left(1 - \frac{2}{3}ik - \frac{k^2}{2F^2}\right)^{1/2}}_{p + iq}$$

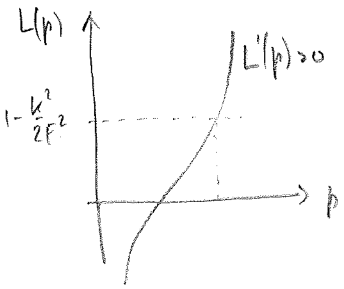
$(p + iq)^2 = 1 - \frac{2}{3}ik - \frac{k^2}{2F^2}$ $\quad (p > 0 \text{ w.l.o.g.})$

$\Rightarrow pq = -\frac{1}{3}k \quad \rightarrow \boxed{q = -k/3p}$

& $p^2 - q^2 = 1 - \frac{k^2}{2F^2} \quad \rightarrow \boxed{p^2 - \frac{k^2}{9p^2} = 1 - \frac{k^2}{2F^2}}$

$$\underbrace{p^2 - \frac{k^2}{9p^2}}_{= L(p)}$$

$$\Rightarrow \boxed{\sigma = -1 \pm p - ik\left(1 \pm \tfrac{1}{3p}\right)}$$



Velocity of perturbations is $-\frac{\sigma_I}{k} = 1 \pm \frac{1}{3p}$.

propagate up and downstream if $p < \frac{1}{3} \Leftrightarrow L(p) < L(\frac{1}{3})$

$\Leftrightarrow 1 - \frac{k^2}{2F^2} < \frac{1}{9} - k^2$

$\Leftrightarrow k^2\left(\frac{1}{2F^2} - 1\right) > \frac{8}{9}$

This occurs for at least some k, if $F^2 < \frac{1}{2}$. i.e. $\boxed{F < \frac{1}{\sqrt{2}} = F_1}$

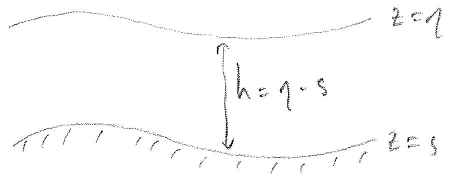
Otherwise $p > \frac{1}{3}$ and all waves propagate downstream.

Instability occurs if $\sigma_R > 0 \Leftrightarrow p > 1 \Leftrightarrow L(p) > L(1)$

$\Leftrightarrow 1 - \frac{k^2}{2F^2} > 1 - \frac{k^2}{9}$

$\Leftrightarrow F^2 > \frac{9}{2}$ $\quad$ i.e. $\boxed{F > \frac{3}{\sqrt{2}} = F_2}$

3. Dunes and antidunes



(i) $\varepsilon h_t + (hu)_x = 0$ Mass conservation

$F^2(\varepsilon u_t + u u_x) = -\eta_x + \delta\left(1-\frac{u^2}{h}\right)$ Momentum conservation. (terms on right: pressure gradient, downslope gravity, friction)

$h(\varepsilon c_t + u c_x) = E(u) - c = -s_t.$ Suspended sediment conservation + Exner equation with erosion rate $E(u)$, deposition rate c (no bedload transport)

F is the Froude number, describes how fast the flow is.
ε represents ratio of advection timescale (for water) to bed evolution timescale.
δ represents importance of gravity and friction to force balance on the scale of a dune.

(ii) Steady state $u=h=c=1$, $s=0$ perturbed by writing $s=S$, $u=1+U$, $h=1+H$, $c=1+C$, then with $\varepsilon=0$,

$$H+U=0 \qquad \rightarrow \quad H=-U$$

$$F^2 U_x = -H_x - S_x + \delta(H-2U) \qquad (F^2-1)U_x + 3\delta U = -S_x$$

$$C_x = \cancel{E(1)} - \cancel{1} + E'(1)U - C = -S_t \qquad C_x = E'(1)U - C = -S_t.$$

Suppose $U = e^{\sigma t+ikx}$ $C=\hat{C}e^{\sigma t+ikx}$ $S=\hat{S}e^{\sigma t+ikx}$. Then,

$$ik\hat{C} = E'(1) - \hat{C} \quad \rightarrow \quad \hat{C} = \frac{E'(1)}{1+ik}$$

$$ik(F^2-1) + 3\delta = -ik\hat{S} \quad \rightarrow -\hat{S} = \frac{ik(F^2-1)+3\delta}{ik}$$

$$ik\hat{C} = -\sigma\hat{S}$$

$$\Rightarrow \sigma = \frac{ik\hat{C}}{-\hat{S}} = \frac{E'(1)\,ik}{(1+ik)\left((F^2-1)-3i\delta/k\right)}$$

$$= \frac{E'(1)\,ik(1-ik)\left((F^2-1)+3i\delta/k\right)}{(1+k^2)\left[(F^2-1)^2+9\delta^2/k^2\right]}$$

$$= \frac{E'(1)\left\{\left[(F^2-1)k^2-3\delta\right]+ik\left[(F^2-1)+3\delta\right]\right\}}{(1+k^2)\left[(F^2-1)^2+9\delta^2/k^2\right]}.$$

If $F>1$, $\mathrm{Re}\,\sigma>0$ for $k^2>\frac{3\delta}{F^2-1}$ so unstable.
$-\mathrm{Im}\frac{\sigma}{k}<0$ in this case so waves move upstream. } → corresponds to growth of antidunes

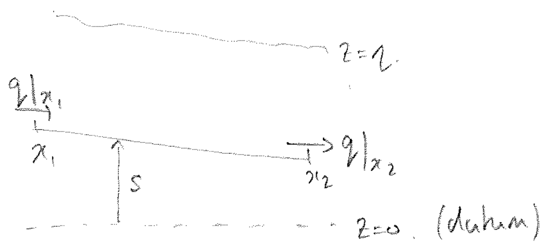
If $F<1$, $\mathrm{Re}\,\sigma<0$ so system is stable.
$-\mathrm{Im}\frac{\sigma}{k}$ has sign depending on δ and F.

(iii) Note $\eta-1 = H+S = (\hat{S}-1)e^{\sigma t+ikx}$, $\frac{\hat{S}-1}{\hat{S}} = 1+\frac{1}{(F^2-1)-3i\delta/k} = 1+\frac{(F^2-1)+\frac{3i\delta}{k}}{(F^2-1)^2+\frac{9\delta^2}{k^2}}$

$= \frac{\hat{S}-1}{\hat{S}}S.$

The argument of this factor gives the phase difference between surface and bed.
If $\boxed{\frac{\delta}{k}\ll 1}$, this is $\boxed{\frac{F^2}{F^2-1}}$, so suggests in phase if $F>1$, out of phase if $F<1$. If $\boxed{\frac{\delta}{k}\gg 1}$ it is $\boxed{1+\frac{ik}{3\delta}+\ldots\approx 1}$
so waves are in phase.

4. Eddy-viscosity model



(i) Consider a section of the bed between x_1 and x_2.

$$\frac{\partial}{\partial t}\left(\int_{x_1}^{x_2}(1-n)S\,dx\right) = q|_{x_1} - q|_{x_2} = -\int_{x_1}^{x_2}\frac{\partial q}{\partial x}\,dx$$

rate of change of volume of sediments in the section.

Since x_1, x_2 are arbitrary, $\boxed{(1-n)\frac{\partial s}{\partial t}+\frac{\partial q}{\partial x}=0}$ assume porosity n is constant.

If $q=q(\tau)$. $(1-n)\frac{\partial s}{\partial t}+q'(\tau)\frac{\partial \tau}{\partial x}=0$

and if we scale ~~to~~ $[t]$ such that $[t]=\frac{(1-n)[s][x]}{[q]}$ the equation becomes ← scales for $[s]$, $[x]$, $[q]$ all arbitrary at this stage.

$$\boxed{\frac{\partial s}{\partial t}+q'(\tau)\frac{\partial \tau}{\partial x}=0.}$$

(ii) Dimensionless shear stress model: $\tau = 1-s+\int_0^\infty k(\xi)\frac{\partial s}{\partial x}(x-\xi,t)\,d\xi$ for $k(\xi)=\frac{\mu}{\xi^{1/3}}$.

Perturb the steady state $s=0+S$, $\tau=1+T$, then

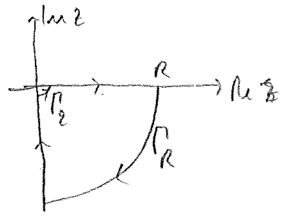
$$\boxed{\frac{\partial S}{\partial t}+q'(1)\frac{\partial T}{\partial x}=0.} \qquad \boxed{T=-S+\int_0^\infty \frac{\mu}{\xi^{1/3}}\frac{\partial S}{\partial x}(x-\xi,t)\,d\xi}$$

Look for solutions of form $S=e^{\sigma t+ikx}$ $T=\hat{T}e^{\sigma t+ikx}$, then

$$\boxed{\sigma=-ikq'(1)\hat{T}}, \quad \boxed{\hat{T}=-1+ik\int_0^\infty \frac{\mu}{\xi^{1/3}}e^{-ik\xi}\,d\xi.} = -1+\mu\underbrace{e^{-i\pi/3}}_{\frac{1}{2}-\frac{\sqrt{3}}{2}i}ik^{1/3}\Gamma\left(\tfrac{2}{3}\right) \quad \text{(using hint).}$$

$$\int_0^\infty \frac{1}{\xi^{1/3}}e^{-ik\xi}\,d\xi = \lim_{\substack{R\to\infty\\ \varepsilon\to 0}}\Bigg(-\underbrace{\int_\Gamma z^{-1/3}e^{-ikz}\,dz}_{0\text{ by Cauchy's theorem}}+\underbrace{\int_{\Gamma_R} z^{-1/3}e^{-ikz}\,dz}_{\to 0\text{ by Jordan's lemma}}+\int_0^{-iR} z^{-1/3}e^{-ikz}\,dz-\underbrace{\int_{\Gamma_\varepsilon} z^{-1/3}e^{-ikz}\,dz}_{\to 0\text{ by estimation lemma}}\Bigg)$$

$z=-\frac{it}{k}$



$$= -0+0+\left(-\frac{i}{k}\right)^{2/3}\underbrace{\int_0^\infty t^{-1/3}e^{-t}\,dt}_{\Gamma(\frac{2}{3})}-0 \quad \text{(assuming } k>0\text{).}$$

$$= \frac{e^{-i\pi/3}}{k^{2/3}}\Gamma\left(\tfrac{2}{3}\right)$$

So $\boxed{\sigma=\frac{1}{2}\mu q'(1)\Gamma\left(\tfrac{2}{3}\right)k^{4/3}-ikq'(1)\left[\frac{\sqrt{3}}{2}\mu q'(1)\Gamma\left(\tfrac{2}{3}\right)k^{4/3}-1\right]}$

Perturbations are unstable ($\sigma_R>0$) assuming $q'(1)>0$. Perturbations travel downstream ($\frac{-\sigma_I}{k}>0$) for large k and upstream for small k (long wavelength).