DMAC 2B68
DMCO 2B68

SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part C: Paper C6.3b

Honour School of Mathematics and Computer Science Part C: Paper C6.3b

APPLIED COMPLEX VARIABLES

Trinity Term 2011

Thursday, 9 June 2011, 9.30am to 11.00am

You may attempt as many questions as you like but **only your two best answers will count.**

Start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

Do not turn this page until you are told that you may do so

1. (a) (i) The domain D in the ζ-plane is bounded by a polygon with exterior angles $\beta_j\pi$, $j = 1, \ldots, n$. The conformal map $\zeta = f(z)$ maps the upper half-plane $\mathrm{Im}(z) > 0$ onto D, with the finite points $x_1 < x_2 < \cdots < x_n$ on the real axis being mapped to the vertices of the polygon. Verify the Schwarz–Christoffel formula

$$\frac{\mathrm{d}f}{\mathrm{d}z} = c\prod_{j=1}^{n}(z - x_j)^{-\beta_j},$$

where c is a constant. In general, how many of the x_j can be specified independently? How is the formula modified if $x_n = \infty$?

(ii) When D consists of the upper half plane $\mathrm{Im}(\zeta) > 0$ with the line segment from $\zeta = 0$ to $\zeta = \mathrm{i}$ removed, and the points $x_1 = -1$, x_2, $x_3 = 1$ and $x_4 = \infty$ are mapped by $\zeta = f(z)$ to $\zeta_1 = 0$, $\zeta_2 = i$, $\zeta_3 = 0$ and $\zeta_4 = \infty$ respectively, show that $x_2 = 0$ and $f(z) = (z^2 - 1)^{1/2}$.

(b) Inviscid irrotational fluid flows steadily in the domain Ω shown in the diagram below, between a rigid wall ABC consisting of two semi-infinite straight line segments meeting at $45°$, and a free surface $A'C'$. The fluid layer has thickness 1 and velocity $(1, 0)$ far downstream, at CC'. The complex potential is $w(z) = \phi + \mathrm{i}\psi$, which satisfies the conditions

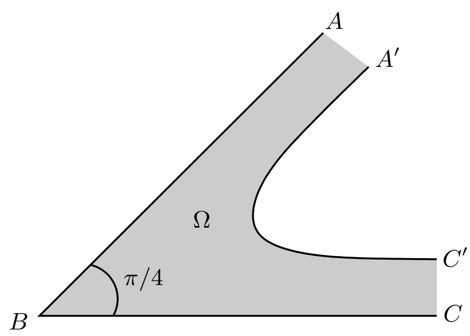
$$\psi = 0 \text{ on } ABC, \qquad \psi = 1, \quad |w'| = 1 \text{ on } A'C',$$

where $w'(z) = u - \mathrm{i}v$ is the complex velocity. In addition, $\phi = 0$ at B.

(i) Sketch the potential (w) and hodograph (w') planes.

(ii) Use appropriate conformal maps to show that

$$\mathrm{e}^{\pi w} = \left(\frac{(w')^{4/3} + 1}{(w')^{4/3} - 1}\right)^2.$$



2. (a) The velocity potential $\phi(x, y, t)$ in two-dimensional unsteady Darcy flow satisfies $\nabla^2\phi = 0$ in a finite domain $D(t)$ on whose boundary $\partial D(t)$ the two conditions

$$\phi = 0, \quad \frac{\partial \phi}{\partial t} + |\nabla \phi|^2 = 0$$

are both satisfied. The flow is driven by a point source of constant strength $2\pi Q$ at the origin, with $\phi \sim \mathrm{Re}\,(Q \log z)$ as $z = x + \mathrm{i}y \to 0$. The map $z = F(\zeta, t)$ takes the unit disc $|\zeta| < 1$ to $D(t)$, with $\zeta = 0$ mapping to the source. Show that $\phi = \mathrm{Re}\,(Q \log \zeta(z, t))$ and deduce that

$$\mathrm{Re}\left(\zeta \frac{\partial F}{\partial \zeta} \overline{\frac{\partial F}{\partial t}}\right) = Q \quad \text{on } |\zeta| = 1.$$

(b) Show that if Γ is a contour in the complex plane and

$$w(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z},$$

then, if f is continuous on Γ, t is any point at which Γ is smooth and f is holomorphic in a neighbourhood of t, the limiting values of $w(z)$ as Γ is approached from either side are $w_\pm(t)$, where

$$w_\pm(t) = \pm\frac{1}{2} f(t) + \frac{1}{2\pi \mathrm{i}} ⨍_\Gamma \frac{f(\zeta)\mathrm{d}\zeta}{\zeta - t}$$

and you should define the integral $⨍$ precisely.

Let $\Gamma = \{x + \mathrm{i}y : 0 < x < 1,\ y = 0\}$ and $\overline{\Gamma} = \{x + \mathrm{i}y : 0 \le x \le 1,\ y = 0\}$. Suppose $w(z)$ is holomorphic away from $\overline{\Gamma}$ and $w_+(x) + w_-(x) = G(x)$ on Γ for some known smooth complex-valued function $G(x)$. Suppose $\widetilde{w}(z)$ is holomorphic and non-zero away from $\overline{\Gamma}$ and $\widetilde{w}_+(x) = -\widetilde{w}_-(x) \neq 0$ on Γ. Determine the density $F(\xi)$ for which a solution for $w(z)$ is given by

$$\frac{w(z)}{\widetilde{w}(z)} = \frac{1}{2\pi \mathrm{i}} \int_0^1 \frac{F(\xi)\mathrm{d}\xi}{\xi - z}.$$

Deduce that

$$f(x) = \frac{\widetilde{w}_+(x)}{\pi \mathrm{i}} ⨍_0^1 \frac{G(\xi)\,\mathrm{d}\xi}{\widetilde{w}_+(\xi)(\xi - x)}$$

is a solution of the singular integral equation

$$\frac{1}{\pi \mathrm{i}} ⨍_0^1 \frac{f(\xi)\,\mathrm{d}\xi}{\xi - x} = G(x) \qquad \text{for} \quad 0 < x < 1.$$

3. (a) Show that the ordinary differential equation

$$\frac{\mathrm{d}^3 w}{\mathrm{d}z^3} + zw = 0$$

has a non-trivial solution of the form

$$w(z) = \int_\Gamma g(\zeta) \mathrm{e}^{z\zeta} \, \mathrm{d}\zeta$$

for a function g and contour Γ only if $g = A\,\mathrm{e}^{\zeta^4/4}$ and $[g(\zeta)\mathrm{e}^{z\zeta}]_\Gamma = 0$, where A is a constant. Identify three choices of Γ which lead to independent solutions.

(b) The complex-valued function $y(x)$ is piecewise smooth and satisfies

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + a^2 y = 0 \quad \text{for } x > 0, \qquad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + b^2 y = 0 \quad \text{for } x < 0,$$

where $0 < \mathrm{Im}(a) < \mathrm{Im}(b)$, with

$$y(0+) = y(0-), \quad \frac{\mathrm{d}y}{\mathrm{d}x}(0+) - \frac{\mathrm{d}y}{\mathrm{d}x}(0-) = 1$$

and $y = \mathrm{O}(\mathrm{e}^{-\gamma|x|})$ as $|x| \to \infty$, where $\gamma > 0$.

(i) Show that, if

$$\overline{y}_-(k) = \int_{-\infty}^{0} y(x)\mathrm{e}^{\mathrm{i}kx} \, \mathrm{d}x, \quad \overline{y}_+(k) = \int_0^\infty y(x)\mathrm{e}^{\mathrm{i}kx} \, \mathrm{d}x,$$

then

$$\left(\frac{k+a}{k+b}\right) \overline{y}_+(k) + \left(\frac{k-b}{k-a}\right) \overline{y}_-(k) = \frac{1}{a+b} \left(\frac{1}{k+b} - \frac{1}{k-a}\right). \qquad (*)$$

(ii) In which upper-half plane $\mathrm{Im}(k) > \alpha$ is $\overline{y}_+(k)$ holomorphic? In which lower-half plane $\mathrm{Im}(k) < \beta$ is $\overline{y}_-(k)$ holomorphic? In which strip of the complex k-plane is $(*)$ valid?

(iii) Assuming in addition that $\overline{y}_+(k) \to 0$ as $|k| \to \infty$ in $\mathrm{Im}(k) > \alpha$ and $\overline{y}_-(k) \to 0$ as $|k| \to \infty$ in $\mathrm{Im}(k) < \beta$, deduce from (i) that

$$y(x) = \frac{\mathrm{e}^{\mathrm{i}ax}}{\mathrm{i}(a+b)} \quad \text{for } x > 0, \qquad y(x) = \frac{\mathrm{e}^{-\mathrm{i}bx}}{\mathrm{i}(a+b)} \quad \text{for } x < 0,$$

and hence determine γ.

1(a)(i) The tangent to ∂D has direction $\arg f'(z)$ because $dz = dx$ is real on ∂D. Thus we require $\arg f'(z)$ to be constant on each side of ∂D with

$$[\arg f'(z)]_{x_j^-}^{x_j^+} = \beta_j \pi,$$

and in addition that $f'(z) \neq 0$ for $z \neq x_j$. We see that

$$\arg f'(z) = \arg c + \sum_{j=1}^{n} \arg (z - x_j)^{-\beta_j}$$

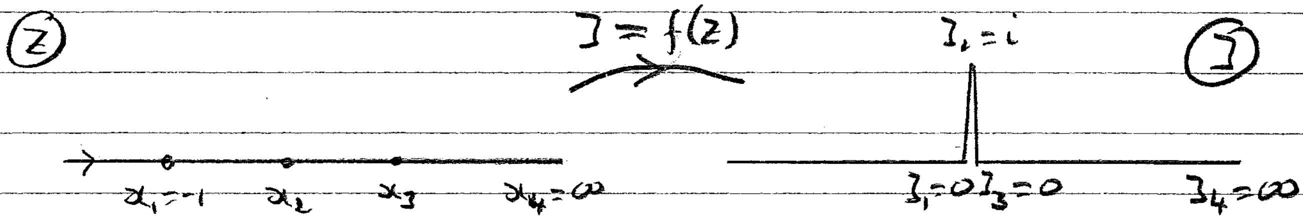
has exactly these properties

In general three of the x_j can be specified independently, by the Riemann Mapping Theorem.

If $x_n = \infty$ the formula is modified to

$$\frac{df}{dz} = c \prod_{j=1}^{n-1} (z - x_j)^{-\beta_j}$$

1(a)(ii)



Exterior angles : $\beta_1 = \frac{1}{2}$, $\beta_2 = -1$, $\beta_3 = \frac{1}{2}$, $\beta_4 = 2$

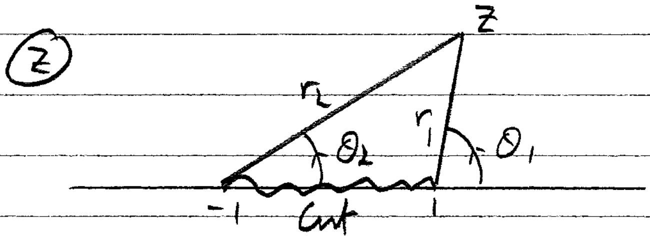
$$\Rightarrow \quad \frac{df}{dz} = c\,(z+1)^{-1/2}(z-x_2)^{1}(z-1)^{-1/2}$$

But symmetry $\Rightarrow \operatorname{Re} f(-x+iy) = -\operatorname{Re} f(x+iy) \Rightarrow x_2 = 0$
(NB: "By symmetry, $x_2 = 0$" will not lose any marks.)

$$\Rightarrow \quad f(z) = a + c\int^{z} \frac{t\,dt}{(t^2-1)^{1/2}} \qquad (a \in \mathbb{C} \text{ arbitrary})$$

$$\Rightarrow \quad f(z) = a + c\,(z^2-1)^{1/2}$$

Define $(z^2-1)^{1/2} = (r_1 r_2)^{1/2} e^{i(\theta_1+\theta_2)/2}$ $(r_1, r_2 > 0;\ -\pi < \theta_1, \theta_2 \le \pi)$



$$\left.\begin{array}{l} 0 = f(\pm 1) = a \\ i = f(0) = c\,(1\cdot 1)^{1/2} e^{i(\pi+0)/2} \end{array}\right] \Rightarrow a = 0,\ c = 1$$

$$\Rightarrow f(z) = (z^2-1)^{1/2}$$

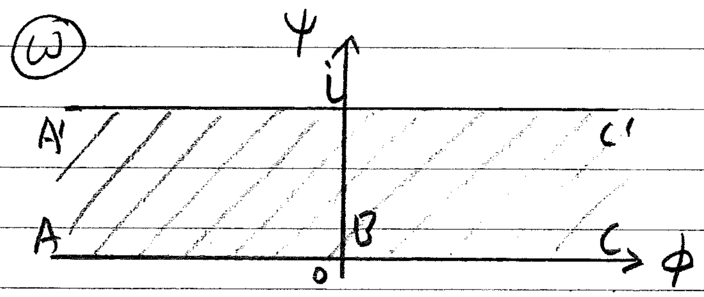
1(b) (i) Potential plane bounded by $\psi = 0$ and $\psi = 1$.

At A, $y = x \to \infty \Rightarrow w' \to \frac{-1+i}{\sqrt{2}} = e^{3\pi i/4} \Rightarrow \phi \to -\infty$

At B, $\phi := 0$

At C, $x \to \infty \Rightarrow w' \to 1 \Rightarrow \phi \to \infty$

$\Rightarrow$



AA' : $w' = e^{3\pi i/4}$

B : $w' = 0$ (stagnation point $w = O(z^{\pi/\alpha})$, $\alpha = \frac{3\pi}{4}$)

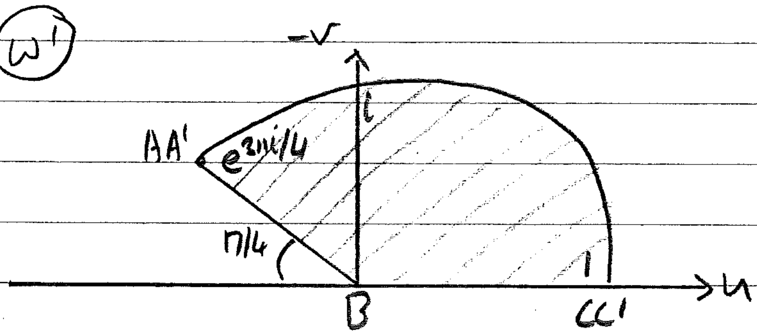
CC' : $w' = 1$

AB : $\arg(w') = 3\pi/4$

BC : $v = 0$

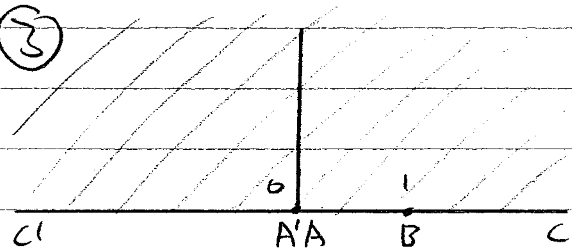
A'C' : $|w'| = 1$

$\Rightarrow$

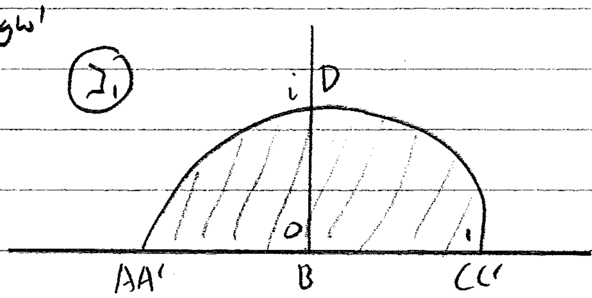


1(b)(ii)

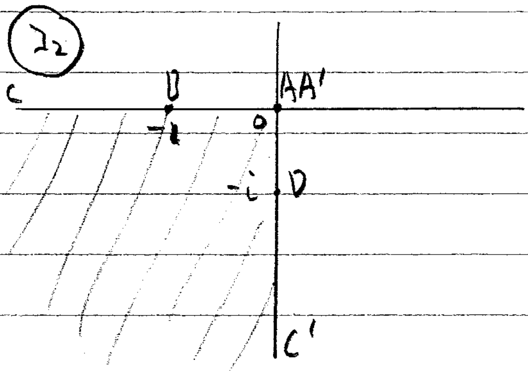
$$w \xrightarrow{z = e^{\pi w}} z$$



$$w' \xrightarrow{z_1 = (w')^{4/3} := e^{\frac{4}{3}\mathrm{Log}\, w'}} z_1$$



$$z_1 \xrightarrow{z_2 = \frac{z_1+1}{z_1-1}} z_2$$



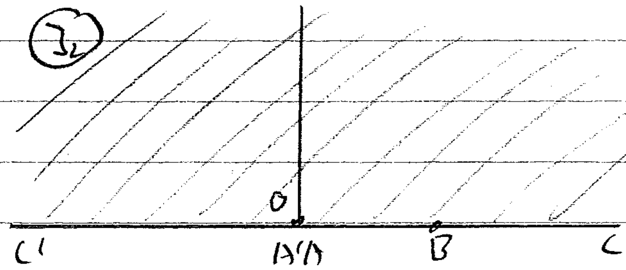
$-1 \to 0$

$0 \to -1$

$1 \to \infty$

$i \to \frac{i+1}{i-1} = -i$

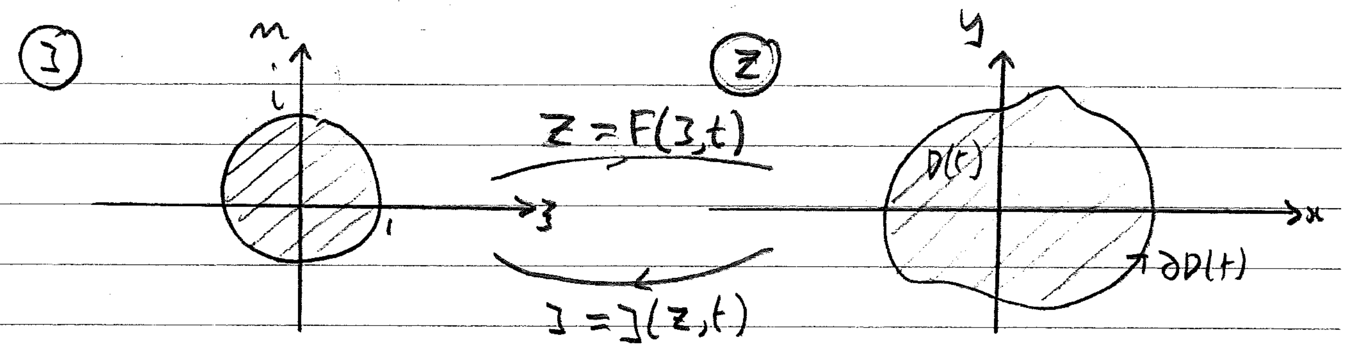
$$z_2 \xrightarrow{z_3 = z_2^2} z_3$$



Points match up in z- and z_3-planes

$$\Rightarrow e^{\pi w} = z = z_3 = z_2^2 = \left(\frac{z_1+1}{z_1-1}\right)^2 = \left(\frac{(w')^{4/3}+1}{(w')^{4/3}-1}\right)^2$$

2(a)



Let $\phi = \mathrm{Re}(w(z,t))$, $W(\zeta,t) = w(F(\zeta,t),t)$, $\Phi = \mathrm{Re}(W(\zeta,t))$

$\nabla^2\phi = 0$ in $D(t)\setminus\{(0,0)\}$ $\Rightarrow$ $\nabla^2\Phi = 0$ in $0<|\zeta|<1$

$\phi \sim Q\log|z|$ as $z\to 0$ $\Rightarrow$ $\Phi \sim Q\log|F(0,t)+F'(0,t)\zeta| \sim Q\log|\zeta|$ as $\zeta\to 0$ $\because$ $F(0,t)=0$

$\phi = 0$ on ∂D $\Rightarrow$ $\Phi = 0$ on $|\zeta| = 1$

Hence, $\phi(x,y,t) = \Phi(\xi,\eta,t) = Q\log|\zeta| = \mathrm{Re}(Q\log\zeta(z,t))$ and this is the unique solution.

$F(\zeta(z,t),t) = z$ $\Rightarrow$ $F_\zeta\,\zeta_t + F_t = 0$

$w(z,t) = Q\log\zeta(z,t)$ $\Rightarrow$ $\phi_t = \mathrm{Re}(w_t) = \mathrm{Re}\left(\frac{Q}{\zeta}\zeta_t\right)$

Also $|\nabla\phi|^2 = \left|\frac{\partial w}{\partial z}\right|^2 = \left|\frac{\partial W}{\partial \zeta}\right|^2 \Big/ \left|\frac{\partial z}{\partial \zeta}\right|^2$

Hence $\phi_t + |\nabla\phi|^2 = 0$ on $\partial D(t)$

$\Rightarrow$ $\mathrm{Re}\left(\frac{Q}{\zeta}\zeta_t\right) + \left|\frac{Q}{\zeta}\right|^2 \Big/ |F_\zeta|^2 = 0$ on $|\zeta| = 1$

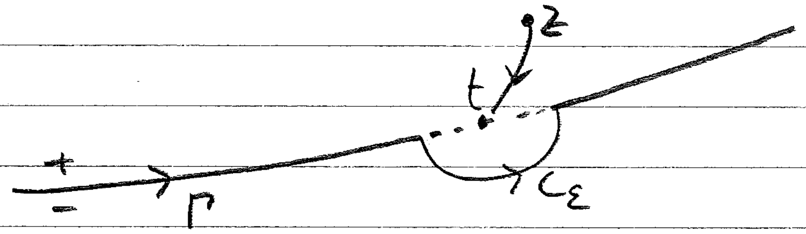
$\Rightarrow$ $\mathrm{Re}\left(-\frac{Q}{\zeta}\frac{F_t}{F_\zeta}\zeta\bar{\zeta}F_\zeta\bar{F}_\zeta\right) + Q^2 = 0$ on $|\zeta| = 1$

$\Rightarrow$ $\mathrm{Re}(\bar{\zeta}\bar{F}_\zeta F_t) = Q$ on $|\zeta| = 1$

$\Rightarrow$ $\mathrm{Re}(\zeta F_\zeta \bar{F}_t) = Q$ on $|\zeta| = 1$

2(b) Label the LHS of Γ as "+" and the RHS as "−".

As $z \to t \in \Gamma$ from the plus side indent Γ with a small (approximate) semi-circle C_ε around t as shown, where the radius ε is sufficiently small that f is holomorphic in $D(t, 2\varepsilon) := \{z : |z-t| < 2\varepsilon\}$.



Let $\gamma_\varepsilon = \Gamma \cap D(t,\varepsilon)$, i.e. the portion of Γ replaced by C_ε.

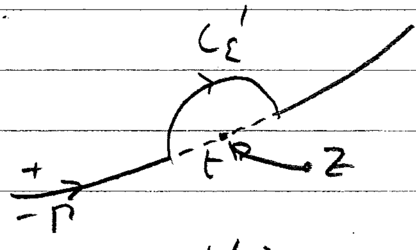
Deformation thm $\Rightarrow$ $$w(z) = \frac{1}{2\pi i}\left(\int_{\Gamma\setminus\gamma_\varepsilon} + \int_{C_\varepsilon}\right)\frac{f(\zeta)}{\zeta - z}\,d\zeta$$

$$\underset{z\to t}{\Rightarrow} \quad w_+(t) = \frac{1}{2\pi i}\left(\int_{\Gamma\setminus\gamma_\varepsilon} + \int_{C_\varepsilon}\right)\frac{f(\zeta)}{\zeta - t}\,d\zeta$$

$$\underset{\varepsilon\to 0}{\Rightarrow} \quad w_+(t) = \underbrace{\frac{1}{2\pi i}⨍_{\Gamma}\frac{f(\zeta)}{\zeta - t}\,d\zeta}_{⨍_\Gamma := \lim_{\varepsilon\to 0}\int_{\Gamma\setminus\gamma_\varepsilon}} + \underbrace{\frac{1}{2}\cdot 2\pi i \cdot \operatorname{res}_{\zeta = t}\frac{f(\zeta)}{2\pi i(\zeta - t)}}_{=\frac{1}{2}f(t)}$$

The principal value integral exists $\because$ log singularities cancel as $\varepsilon \to 0$ by cty.

For $w_-(t)$ replace γ_ε with C_ε' as shown:



Semi-circle now gives a contribution $-\frac{1}{2}\cdot 2\pi i \cdot \operatorname{res}_{\zeta=t}\frac{f(\zeta)}{2\pi i(\zeta - t)}$

Hence $$w_\pm(t) = \pm\frac{1}{2}f(t) + \frac{1}{2\pi i}⨍_{\Gamma}\frac{f(\zeta)}{\zeta - t}\,d\zeta$$

2(b) Let $w(z) = \frac{1}{2\pi i}\int_0^1 \frac{f(\zeta)}{\zeta - z}\,d\zeta$

Plemelj $\Rightarrow$ $w_+ - w_- = f$, $w_+ + w_- = \frac{1}{\pi i}⨍_0^1 \frac{f(\zeta)}{\zeta - x}\,d\zeta$ on Γ

Let $W(z) = \frac{w(z)}{\tilde{w}(z)} = \frac{1}{2\pi i}\int_0^1 \frac{F(\zeta)}{\zeta - z}\,d\zeta$

Plemelj $\Rightarrow$ $W_+ - W_- = F$, $W_+ + W_- = \frac{1}{\pi i}⨍_0^1 \frac{F(\zeta)}{\zeta - x}\,d\zeta$ on Γ

(i) $F = W_+ - W_-$

$$= \frac{w_+}{\tilde{w}_+} - \frac{w_-}{\tilde{w}_-}$$

$$= \frac{w_+ + w_-}{\tilde{w}_+} \qquad (\because \tilde{w}_- = -\tilde{w}_+ \text{ on } \Gamma)$$

$$= \frac{G}{\tilde{w}_+} \qquad (\because w_+ + w_- = G \text{ on } \Gamma)$$

(ii) $W_+ + W_- = \frac{w_+}{\tilde{w}_+} + \frac{w_-}{\tilde{w}_-} = \frac{w_+ - w_-}{\tilde{w}_+} = \frac{f}{\tilde{w}_+}$

$$\Rightarrow f(x) = \tilde{w}_+(x)\left(W_+(x) + W_-(x)\right)$$

$$= \frac{\tilde{w}_+(x)}{\pi i}⨍_0^1 \frac{F(\zeta)}{\zeta - x}\,d\zeta$$

$$= \frac{\tilde{w}_+(x)}{\pi i}\int_0^1 \frac{G(\zeta)}{\tilde{w}_+(\zeta)(\zeta - x)}\,d\zeta$$

satisfies $\frac{1}{\pi i}⨍_0^1 \frac{f(\zeta)}{\zeta - x}\,d\zeta = w_+ + w_- = g(x) \quad (0 < x < 1)$

3(a) Differentiating under the integral sign

$$\frac{d^3w}{dz^3} + zw = \int_\Gamma \zeta^3 g(\zeta)e^{z\zeta} + z\,g(\zeta)e^{z\zeta}\,d\zeta$$

$$= \int_\Gamma \frac{\partial}{\partial\zeta}\left(g(\zeta)e^{z\zeta}\right) + \left(\zeta^3 g(\zeta) - g'(\zeta)\right)e^{z\zeta}\,d\zeta$$

$$= \left[g(\zeta)e^{z\zeta}\right]_\Gamma + \int_\Gamma \left(\zeta^3 g(\zeta) - g'(\zeta)\right)e^{z\zeta}\,d\zeta$$

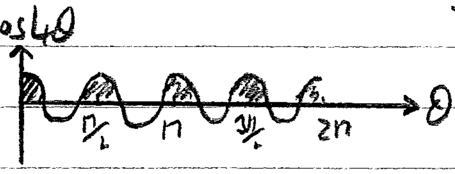
Hence, ODE satisfied only if $\left[g(\zeta)e^{z\zeta}\right]_\Gamma = 0$

and $g'(\zeta) = \zeta^3 g(\zeta)$, i.e. $\frac{g'}{g} = \zeta^3 \Rightarrow g(\zeta) = Ae^{\zeta^4/4}$ $(A \in \mathbb{C}\setminus\{0\})$

Since $\left[g(\zeta)e^{z\zeta}\right]_\Gamma = \left[Ae^{\zeta^4/4 + z\zeta}\right]_\Gamma = 0$, the

contour Γ needs to be s.t. $\mathrm{Re}(\zeta^4) < 0$ at infinity.

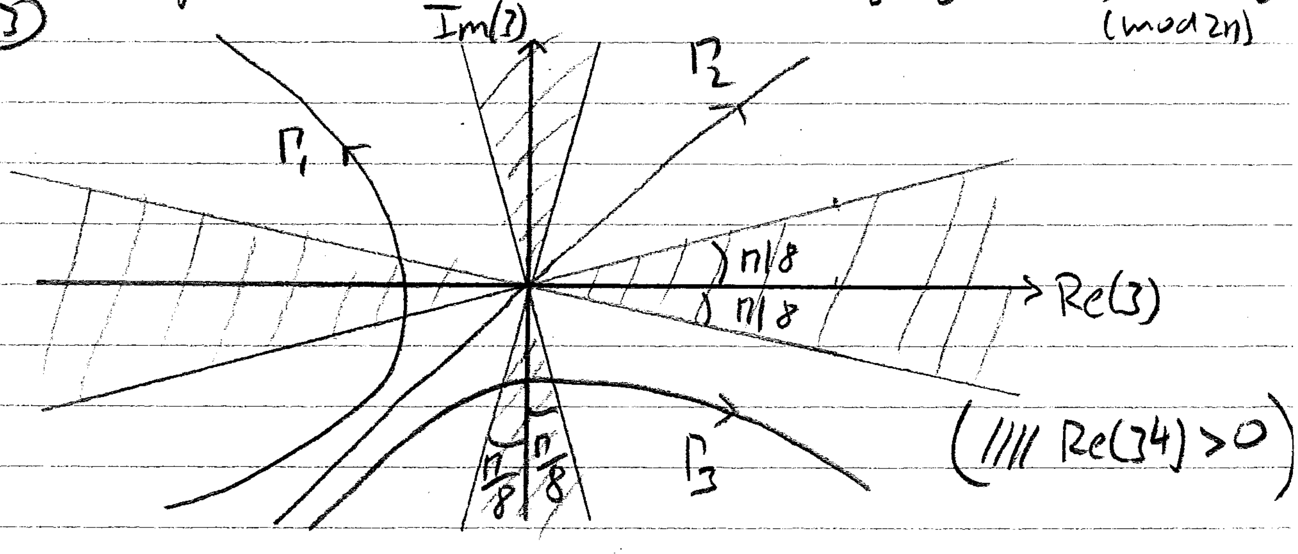
$\zeta = re^{i\theta} \Rightarrow \cos 4\theta < 0$



$\Rightarrow \theta \in \left(\frac{\pi}{8}, \frac{3\pi}{8}\right), +\frac{\pi}{2}, +\pi, +\frac{3\pi}{2}$ (mod 2π)

Shaded segments s.t. $\cos 4\theta > 0 \Rightarrow \theta \in \left(-\frac{\pi}{8}, \frac{\pi}{8}\right), +\frac{\pi}{2}, +\pi, +\frac{3\pi}{2}$ (mod 2π)

(3)



3(b)(i) $0 = \int_0^\infty y'' e^{ikx} + a^2 y e^{ikx} \, dx$

$= a^2 \bar{y}_+ + y' e^{ikx} \Big|_0^\infty - \int_0^\infty y' ik e^{ikx} dx$

$= a^2 \bar{y}_+ - y'(0+) - y ik e^{ikx} \Big|_0^\infty + \int_0^\infty y (ik)^2 e^{ikx} dx$

$= (a^2 - k^2) \bar{y}_+ - y'(0+) + iky(0+)$

provided $\bar{y}_+$ exists and $y ik e^{ikx}$, $y' e^{ikx} \to 0$ as $x \to \infty$, which is the case for $\mathrm{Im}(k) > -\gamma$ because y is piecewise smooth and $y(x) = O(e^{-\gamma x})$ as $x \to \infty$.

Similarly

$$0 = (b^2 - k^2) \bar{y}_- + y'(0-) - iky(0+)$$

provided $\mathrm{Im}(k) < \gamma$.

Hence

$$(a^2 - k^2)\bar{y}_+ + (b^2 - k^2)\bar{y}_- = [y']_{0-}^{0+} - ik[y]_{0-}^{0+}$$

$$\Rightarrow (k-a)(k+a)\bar{y}_+ + (k-b)(k+b)\bar{y}_- = -1$$

$$\Rightarrow \left(\frac{k+a}{k+b}\right)\bar{y}_+ + \left(\frac{k-b}{k-a}\right)\bar{y}_- = \frac{-1}{(k-a)(k+b)} = \frac{1}{a+b}\left(\frac{1}{k+b} - \frac{1}{k-a}\right) \quad (1)$$

provided $\bar{y}_\pm$ exist, i.e. $|\mathrm{Im}(k)| < \gamma$, and $k \neq a, -b$.

3(b)(ii) $\alpha = -\gamma$, $\beta = \gamma$. Since $0 < \mathrm{Im}(a) < \mathrm{Im}(b)$, (1) is valid in the strip $\Omega = \{k : |\mathrm{Im}(k)| < \delta\}$ where $\delta = \min(\gamma, \mathrm{Im}(a)) > 0$, say

3(b)(iii) Define

$$E(k) = \overbrace{\left(\frac{k+a}{k+b}\right)\bar{y}_+ - \frac{1}{(a+b)(k+b)}}^{\oplus} = \overbrace{-\left(\frac{k-b}{k-a}\right)\bar{y}_- - \frac{1}{(a+b)(k-a)}}^{\ominus} \quad \text{on } \Omega$$

and analytically continue $E(k)$ into $\mathbb{C} \setminus \Omega$

$$\Rightarrow \quad E(k) = \begin{cases} \left(\frac{k+a}{k+b}\right)\bar{y}_+ - \frac{1}{(a+b)(k+b)}, & \mathrm{Im}(k) > -\delta \\ -\left(\frac{k-b}{k-a}\right)\bar{y}_- - \frac{1}{(a+b)(k-a)}, & \mathrm{Im}(k) < \delta \end{cases}$$

by the identity theorem and 3(b)(ii) $\Rightarrow$ $\oplus$ term holomorphic in $\mathrm{Im}(k) > -\delta$ and $\ominus$ term holomorphic in $\mathrm{Im}(k) < \delta$.

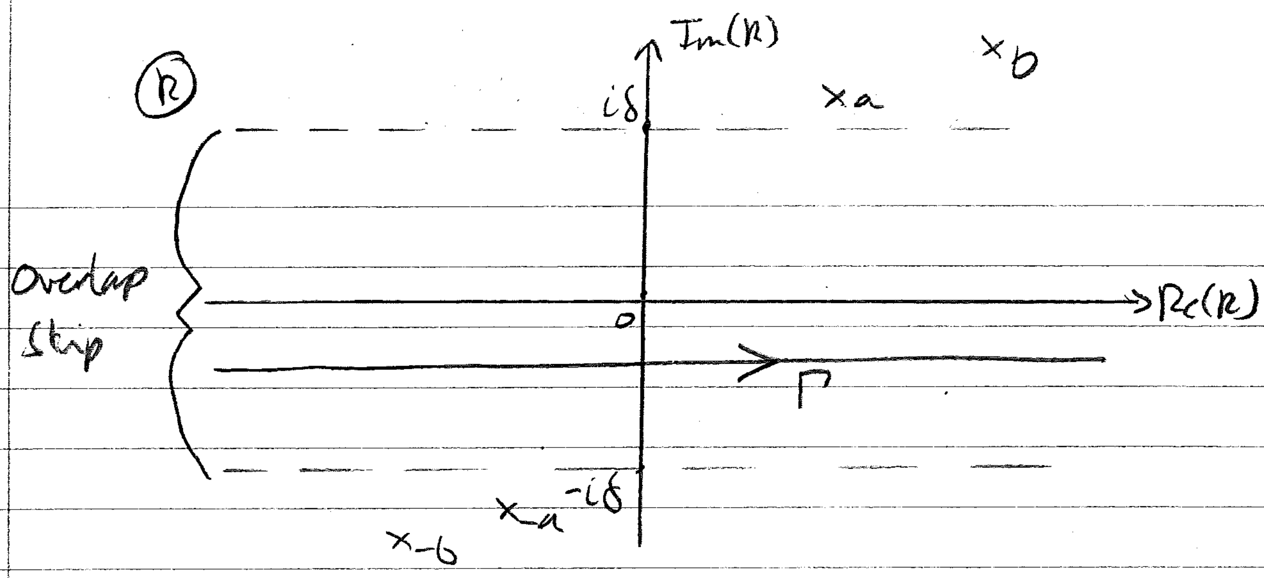
Hence, $E(k)$ is entire and tends to zero at ∞ ($\bar{y}_+(k) \to 0$ at ∞ $\Rightarrow$ $\oplus$ term $\to 0$ as $|k| \to \infty$ in $\mathrm{Im}(k) > -\delta$ and $\ominus$ term $\to 0$ as $|k| \to \infty$ in $\mathrm{Im}(k) < \delta$), so by Liouville $E = 0$.

$$\Rightarrow \quad \bar{y}_+ = \frac{1}{(a+b)(k+a)}, \quad \bar{y}_- = -\frac{1}{(a+b)(k-b)}$$

$$\Rightarrow \quad \text{For } x > 0, \; y(x) = \frac{1}{2\pi} \int_P \frac{e^{-ikx}}{(a+b)(k+a)} \, dk$$

$$\text{For } x < 0, \; y(x) = \frac{-1}{2\pi} \int_P \frac{e^{-ikx}}{(a+b)(k-b)} \, dk$$

where inversion contour P is in overlap strip $|\mathrm{Im}(k)| < \delta$



For $x>0$ close Γ at $-i\infty$ $\Rightarrow$ $y(x) = \frac{-1}{2\pi} \cdot 2\pi i \cdot \mathrm{Res}_{k=-a} \frac{e^{-ikx}}{(a+b)(k+a)}$

$$= \frac{-i e^{iax}}{a+b}$$

$$= \frac{e^{iax}}{i(a+b)}$$

by the residue theorem ($-$ sign $\because$ clockwise)

For $x<0$ close Γ at $i\infty$ $\Rightarrow$ $y(x) = \frac{1}{2\pi} \cdot 2\pi i \cdot \mathrm{Res}_{k=b} \frac{-e^{-ikx}}{(a+b)(k-b)}$

$$= + \frac{e^{-ibx}}{i(a+b)}$$

by the residue theorem ($+$ sign $\because$ anticlockwise)

Hence, $\gamma \leq \mathrm{Im}(a)$ (give full credit for $\gamma = \mathrm{Im}(a)$).