A12207W1

SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part C: Paper C5.6
Honour School of Mathematics and Statistics Part C: Paper C5.6
Honour School of Mathematical and Theoretical Physics Part C: Paper C5.6
Master of Science in Mathematical and Theoretical Physics: Paper C5.6
Master of Science in Mathematical Sciences: Paper C5.6

APPLIED COMPLEX VARIABLES

TRINITY TERM 2019

THURSDAY, 6 JUNE 2019, 9.30am to 11.15am

You may submit answers to as many questions as you wish but only the best two will count for the total mark. All questions are worth 25 marks.

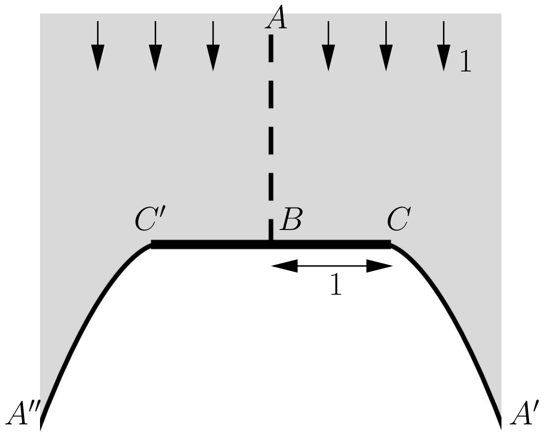
You should ensure that you:

- *start a new answer booklet for each question which you attempt.*
- *indicate on the front page of the answer booklet which question you have attempted in that booklet.*
- *cross out all rough working and any working you do not want to be marked. If you have used separate answer booklets for rough work please cross through the front of each such booklet and attach these answer booklets at the back of your work.*
- *hand in your answers in numerical order.*

If you do not attempt any questions, you should still hand in an answer booklet with the front sheet completed.

Do not turn this page until you are told that you may do so

1. Consider the steady two-dimensional inviscid fluid flow of a uniform stream against a horizontal flat plate illustrated below. The velocity of the incoming flow is $(0, -1)$, and the plate $C'BC$ is of length 2, with B the midpoint of $C'C$. The flow is symmetric about the vertical line AB. Without loss of generality, take the streamfunction ψ and the potential ϕ to be zero at B.



(a) [7 marks] Sketch the images of the right-hand half of the shaded fluid domain (i.e. that bounded by $ABCA'$) in the potential w-plane and the hodograph w'-plane, indicating clearly the image of each of the labelled points.

(b) [6 marks] By conformally mapping the right-hand half of the fluid domain onto the upper half-plane, show that w satisfies the differential equation

$$\frac{a}{a-w} = \left(\frac{(w')^2+1}{(w')^2-1}\right)^2,$$

where a is the value of the potential at the point C.

(c) [6 marks] By integrating $\mathrm{d}z/\mathrm{d}\zeta$ from B to C show that

$$a = \frac{2}{\pi+4}.$$

[*You may use the fact that* $\int_1^\infty \left(\frac{\zeta^{1/2}+1}{\zeta^{1/2}-1}\right)^{1/2} \frac{\mathrm{d}\zeta}{\zeta^2} = (4+\pi)/2$ *without proof.*]

(d) [6 marks] If the velocity on the free surface is given by $w' = \mathrm{e}^{-\mathrm{i}\theta}$, show that CA' is given parametrically by

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{2a\sin\theta}{\cos^2\theta}, \qquad \frac{\mathrm{d}y}{\mathrm{d}\theta} = \frac{2a\sin^2\theta}{\cos^3\theta}.$$

Deduce that $x \to \infty$ as $\theta \to -\pi/2$.

Show that

$$y \sim -\alpha x^2$$

as $x \to \infty$, where α is a constant that you should determine.

2. (a) [5 marks] Let Γ be a smooth contour in the complex plane and

$$w(z) = \frac{1}{2\pi\mathrm{i}} \int_\Gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z},$$

where f is continuous on Γ and holomorphic in a neighbourhood of the point $t \in \Gamma$. Show that the limiting values of $w(z)$ as Γ is approached from either side are $w_\pm(t)$, given by

$$w_\pm(t) = \pm\frac{f(t)}{2} + \frac{1}{2\pi\mathrm{i}} ⨍_\Gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - t},$$

where you should define the integral $⨍$ precisely.

(b) [5 marks] Now let Γ be a closed contour enclosing the origin. Show that

$$\frac{1}{\pi\mathrm{i}} ⨍_\Gamma \frac{\mathrm{d}\zeta}{\zeta - t} = 1$$

and determine

$$\frac{1}{\pi\mathrm{i}} ⨍_\Gamma \frac{\mathrm{d}\zeta}{\zeta(\zeta - t)}.$$

(c) [9 marks] Let $a(z)$ and $b(z)$ be holomorphic in a neighbourhood of the smooth closed contour Γ such that $a(z)^2 - b(z)^2 \neq 0$ for $z \in \Gamma$. By defining $W(z) = (a(z) + b(z))w(z)$ for z inside Γ, and $W(z) = (a(z) - b(z))w(z)$ for z outside Γ, show that a solution of the singular Cauchy integral equation

$$a(t)f(t) + \frac{b(t)}{\pi\mathrm{i}} ⨍_\Gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{(\zeta - t)} = g(t), \qquad t \in \Gamma,$$

is

$$f(t) = \frac{1}{a(t)^2 - b(t)^2}\left(a(t)g(t) - \frac{b(t)}{\pi\mathrm{i}} ⨍_\Gamma \frac{g(\zeta)\,\mathrm{d}\zeta}{\zeta - t}\right).$$

(d) [6 marks] By taking $\int_\Gamma f(\zeta)\,\mathrm{d}\zeta$ to be a constant which you should determine, find a solution to the singular Cauchy integral equation

$$2f(t) + \frac{1}{\pi\mathrm{i}} ⨍_\Gamma \left(\frac{1}{\zeta - t} + \alpha\right) f(\zeta)\,\mathrm{d}\zeta = \frac{1}{t}, \qquad t \in \Gamma,$$

where Γ is a smooth closed contour in the complex plane which encloses the origin.

3. Consider the integral equation

$$\int_0^\infty K(x-t)f(t)\;\mathrm{d}t = 3f(x) \qquad \text{for } x \geqslant 0,$$

where $K(x) = \mathrm{e}^{-|x|} + \mathrm{e}^{-2|x|}$ and you may assume that $f : [0,\infty) \to \mathbb{R}$ is continuous and that $f(x) = O(x)$ as $x \to \infty$.

(a) [5 marks] Show how to convert this to the integral equation

$$\int_{-\infty}^\infty K(x-t)f_+(t)\;\mathrm{d}t = 3f_+(x) + h_-(x)$$

on the full range $-\infty < x < \infty$, and explain how $h_-(x)$ is defined. State for which regions of the complex k-plane the Fourier transforms $\bar{f}_+(k)$ and $\bar{h}_-(k)$ are holomorphic.

(b) [12 marks] Show that

$$\frac{3k^2(k+\sqrt{3}\mathrm{i})}{(k+\mathrm{i})(k+2\mathrm{i})}\bar{f}_+(k) = -\frac{(k-\mathrm{i})(k-2\mathrm{i})}{(k-\sqrt{3}\,\mathrm{i})}\bar{h}_-(k)$$

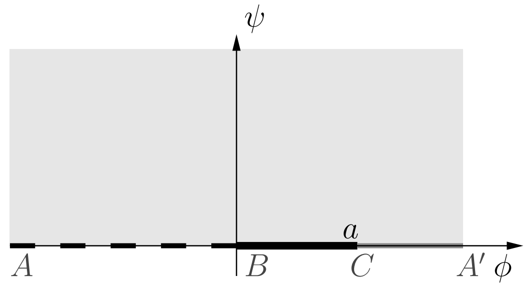
in a strip $\alpha < \mathrm{Im}(k) < \beta$, and give suitable values of α and β. Explain clearly why the left- and right-hand sides of this equation must be constant.
[*You may assume without proof that* $\bar{f}_+(k)$ *and* $\bar{h}_-(k)$ *are* $O(k^{-1})$ *as* $k \to \infty$]

(c) [8 marks] Hence show that

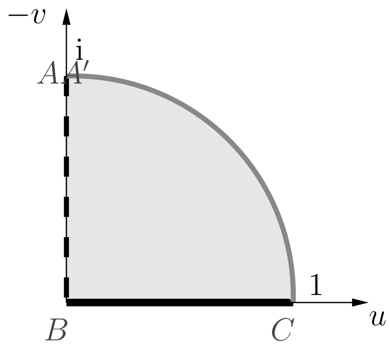
$$f(x) = C\left((5-3\sqrt{3})\mathrm{e}^{-\sqrt{3}\,x} + 3\sqrt{3} - 2 + 2\sqrt{3}\,x\right),$$

where C is an arbitrary constant.

1. (a) [7 marks] Since $\psi = 0$ on $ABCA'$ and $\psi_x = -\phi_y = 1$ at infinity so that $\psi > 0$ in the fluid domain, the potential plane is the upper half plane, where $\phi(C) = a > 0$ is unknown.



The velocity at A and A' is $(0, -1)$. The velocity at B is $(0, 0)$. The velocity at C is $(1, 0)$. On the free surface CA' we have $|w'| = 1$. On AB we know $u = 0$. On BC we know $v = 0$.

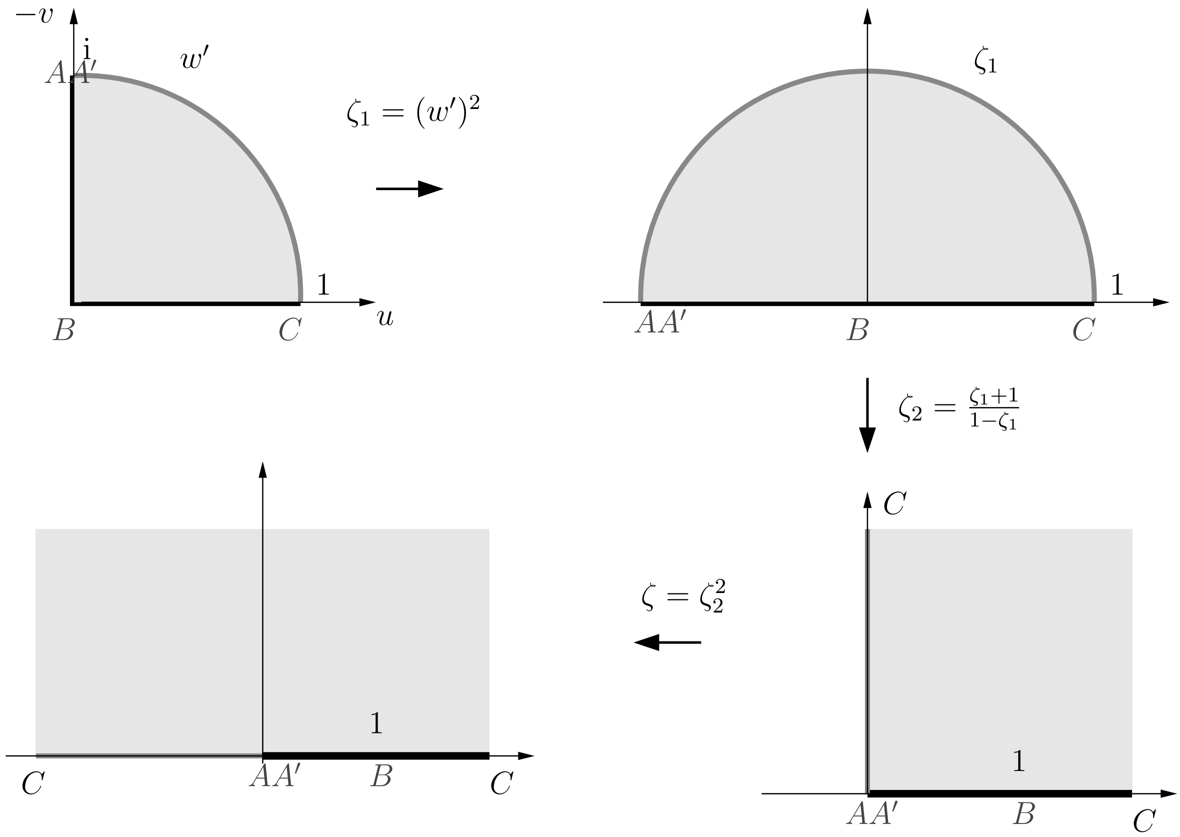


7 **New example**

(b) [6 marks] For the hodograph plane, first get rid of the corner at the origin by squaring: $\zeta_1 = (w')^2$. Then use the mobius map

$$\zeta_2 = \frac{\zeta_1 + 1}{1 - \zeta_1}$$

which sends -1 to 0, 1 to ∞, and 0 to 1. Thus this maps the semicircle to the first quadrant. Now square to get the upper half plane.



Thus

$$\zeta = \left(\frac{(w')^2+1}{(w')^2-1}\right)^2.$$

Although the potential plane is already the upper-half plane, the points are in the wrong places. We need to send infinity to 0, 0 to 1 and a to infinity. Thus we need to set

$$\zeta = \frac{a}{a-w}.$$

Thus w satisfies the differential equation

$$\frac{a}{a-w} = \left(\frac{(w')^2+1}{(w')^2-1}\right)^2.$$

6 New example, but these maps have been seen before

(c) [6 marks] Inverting the hodograph map gives

$$-\zeta^{1/2} = \frac{(w')^2+1}{(w')^2-1} \quad\Rightarrow\quad -\zeta^{1/2}((w')^2-1) = (w')^2+1 \quad\Rightarrow\quad (w')^2(1+\zeta^{1/2}) = \zeta^{1/2}-1$$

$$\Rightarrow\quad w' = \left(\frac{\zeta^{1/2}-1}{\zeta^{1/2}+1}\right)^{1/2}.$$

Note the first minus sign, since ζ is real and positive on BC when w' is real and $0 < w' < 1$. The final plus sign follows by the same argument. Inverting the potential map gives

$$\zeta(a-w) = a \quad\Rightarrow\quad w = a\left(1-\frac{1}{\zeta}\right).$$

By the chain rule

$$\frac{\mathrm{d}z}{\mathrm{d}\zeta} = \frac{1}{w'}\frac{\mathrm{d}w}{\mathrm{d}\zeta} = \left(\frac{\zeta^{1/2}+1}{\zeta^{1/2}-1}\right)^{1/2}\frac{a}{\zeta^2}.$$

Thus

$$1 = \int_B^C \frac{\mathrm{d}z}{\mathrm{d}\zeta}\,\mathrm{d}\zeta = \int_1^\infty \left(\frac{\zeta^{1/2}+1}{\zeta^{1/2}-1}\right)^{1/2}\frac{a}{\zeta^2}\,\mathrm{d}\zeta = \frac{a(4+\pi)}{2}$$

by the hint.

$$a = \frac{2}{\pi+4}.$$

6 **Unseen. Will require some thought**

(d) [6 marks] Setting $w' = \mathrm{e}^{-\mathrm{i}\theta}$ on the free surface gives

$$\frac{a}{a-w} = \left(\frac{\mathrm{e}^{-2\mathrm{i}\theta}+1}{\mathrm{e}^{-2\mathrm{i}\theta}-1}\right)^2 = \left(\frac{\mathrm{e}^{-\mathrm{i}\theta}+\mathrm{e}^{\mathrm{i}\theta}}{\mathrm{e}^{-\mathrm{i}\theta}-\mathrm{e}^{\mathrm{i}\theta}}\right)^2 = -\cot^2\theta.$$

Thus

$$w = \frac{a(1+\cot^2\theta)}{\cot^2\theta} = a\sec^2\theta.$$

Differentiate with respect to θ to give

$$\frac{\mathrm{d}w}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}\theta} = \frac{2a\sin\theta}{\cos^3\theta}.$$

Substituting for $w' = \mathrm{e}^{-\mathrm{i}\theta}$ gives

$$\frac{\mathrm{d}z}{\mathrm{d}\theta} = \frac{2a\mathrm{e}^{\mathrm{i}\theta}\sin\theta}{\cos^3\theta}.$$

Taking the real an imaginary parts gives

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{2a\sin\theta}{\cos^2\theta}, \qquad \frac{\mathrm{d}y}{\mathrm{d}\theta} = \frac{2a\sin^2\theta}{\cos^3\theta}.$$

The boundary conditions are $x(0) = 1$ and $y(0) = 0$. Thus

$$x(\theta) = 1 + \int_0^\theta \frac{\mathrm{d}x}{\mathrm{d}\theta}(\hat\theta)\,\mathrm{d}\hat\theta = 1 + \int_0^\theta \frac{2a\sin\hat\theta}{\cos^2\hat\theta}\,\mathrm{d}\hat\theta = 1 + 2a\left[\frac{1}{\cos\hat\theta}\right]_0^\theta = 1 + 2a(\sec\theta - 1).$$

Thus $x \to \infty$ as $\theta \to -\pi/2$.
With $\theta = -\pi/2 + \phi$, as $\phi \to 0$

$$\frac{\mathrm{d}x}{\mathrm{d}\phi} = -\frac{2a\cos\phi}{\sin^2\phi} \sim -\frac{2a}{\phi^2}, \qquad \frac{\mathrm{d}y}{\mathrm{d}\phi} = \frac{2a\cos^2\phi}{\sin^3\phi} \sim \frac{2a}{\phi^3}.$$

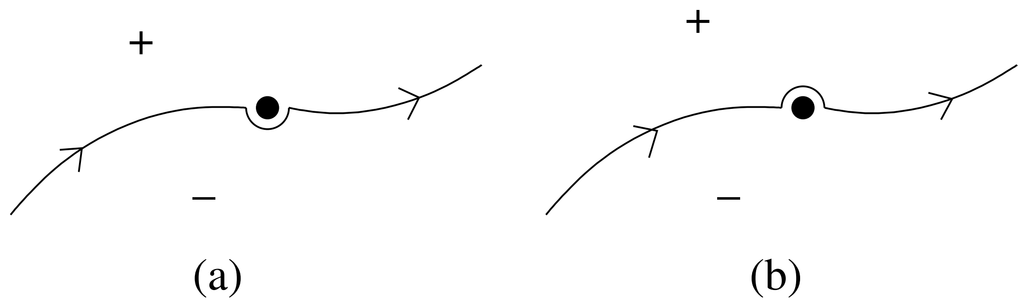
Thus

$$x \sim \frac{2a}{\phi}, \qquad y \sim -\frac{a}{\phi^2} \qquad \Rightarrow \qquad y \sim -\frac{x^2}{4a} = -\frac{(\pi+4)x^2}{8}$$

as x, $-y \to \infty$.

6 **New example, but the method has been seen before. The last bit will require some thought.**

2. (a) [5 marks] Label the left-hand side of Γ as "+" and the right-hand side as "-". As $z \to t \in \Gamma$ from the plus side indent the contour with a small semi-circle around t as shown in (a).



Then

$$w_+(t) = \lim_{\epsilon\to 0}\frac{1}{2\pi\mathrm{i}}\left(\int_{\text{one end}}^{t-\epsilon}\frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - t} + \int_{\text{semicircle}|\zeta-t|=\epsilon}\frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - t} + \int_{\text{other end}}^{t-\epsilon}\frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - t}\right).$$

As $\epsilon \to 0$ the semicircle gives a contribution $\dfrac{1}{2\pi\mathrm{i}} \times \pi\mathrm{i}f(t)$. Thus

$$w_+(t) = \frac{f(t)}{2} + \frac{1}{2\pi\mathrm{i}}\fint_\Gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - t},$$

where

$$\fint = \lim_{\epsilon\to 0}\left(\int^{t-\epsilon} + \int_{t+\epsilon}\right).$$

As $z \to t \in \Gamma$ from the minus side we need to indent the contour on the other side with a small semi-circle around t as shown in (b). The semicircle now gives a contribution $-\dfrac{1}{2\pi\mathrm{i}} \times \pi\mathrm{i}f(t)$ as $\epsilon \to 0$, so that

$$w_-(t) = -\frac{f(t)}{2} + \frac{1}{2\pi\mathrm{i}}\fint_\Gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - t}.$$

[5] **Bookwork**

(b) [5 marks] Consider the case $f(\zeta) = 1$ in part (a). Then, by Cauchy's residue theorem, $w(z) = 1$ when z is inside Γ and $w(z) = 0$ when z is outside Γ. Thus the Plemelj formulae give

$$1 = \frac{1}{2} + \frac{1}{2\pi\mathrm{i}}\fint_\Gamma \frac{\mathrm{d}\zeta}{\zeta - t}, \qquad 0 = -\frac{1}{2} + \frac{1}{2\pi\mathrm{i}}\fint_\Gamma \frac{\mathrm{d}\zeta}{\zeta - t}.$$

Either of these is enough to give

$$\frac{1}{\pi\mathrm{i}}\fint_\Gamma \frac{\mathrm{d}\zeta}{\zeta - t} = 1.$$

Now consider the case $f(\zeta) = 1/\zeta$. Now, by Cauchy's residue theorem, $w(z) = 1/z - 1/z = 0$ when z is inside Γ and $w(z) = -1/z$ when z is outside Γ. Thus the Plemelj formulae give

$$0 = \frac{1}{2t} + \frac{1}{2\pi\mathrm{i}}\fint_\Gamma \frac{\mathrm{d}\zeta}{\zeta(\zeta - t)}, \qquad -\frac{1}{t} = -\frac{1}{2t} + \frac{1}{2\pi\mathrm{i}}\fint_\Gamma \frac{\mathrm{d}\zeta}{\zeta(\zeta - t)}.$$

Either of these is enough to give

$$\frac{1}{\pi\mathrm{i}}\fint_\Gamma \frac{\mathrm{d}\zeta}{\zeta - t} = -\frac{1}{t}.$$

[5] **Unseen. Will require some thought**

(c) [9 marks] With

$$w(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f(\zeta)\ \mathrm{d}\zeta}{\zeta - z}$$

we have, from Plemelj, for $t \in \Gamma$,

$$\frac{1}{\pi \mathrm{i}} \fint_\Gamma \frac{f(\zeta)\ \mathrm{d}\zeta}{\zeta - t} = w_+(t) + w_-(t), \qquad f(t) = w_+(t) - w_-(t).$$

The integral equation is

$$a(t)f(t) + \frac{b(t)}{\pi \mathrm{i}} \fint_\Gamma \frac{f(\zeta)\ \mathrm{d}\zeta}{(\zeta - t)} = g(t), \tag{1}$$

which is therefore

$$(a(t) + b(t))w_+(t) - (a(t) - b(t))w_-(t) = g(t). \tag{2}$$

Define $W(z)$ such that $W(z) = (a(z)+b(z))w(z)$ for z inside Γ and $W(z) = (a(z) - b(z))w(z)$ for z outside Γ. Then we look for a Cauchy integral representation of W in the form

$$W(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{h(\zeta)\ \mathrm{d}\zeta}{\zeta - z},$$

where we need to find $h(\zeta)$. The Plemelj formulae now give

$$W_\pm(t) = \pm\frac{1}{2}h(t) + \frac{1}{2\pi \mathrm{i}} \fint_\Gamma \frac{h(\zeta)\ \mathrm{d}\zeta}{\zeta - t}.$$

Thus

$$h(t) = W_+(t) - W_-(t) = g(t),$$

from (2). Then

$$W_+(t) + W_-(t) = (a+b)w_+ + (a-b)w_- = \frac{1}{\pi \mathrm{i}} \fint_\Gamma \frac{g(\zeta)\ \mathrm{d}\zeta}{\zeta - t},$$

i.e.

$$bf + \frac{a}{\pi \mathrm{i}} \fint_\Gamma \frac{f(\zeta)\ \mathrm{d}\zeta}{(\zeta - t)} = \frac{1}{\pi \mathrm{i}} \fint_\Gamma \frac{g(\zeta)\ \mathrm{d}\zeta}{\zeta - t}, \tag{3}$$

Eliminating $\fint_\Gamma \frac{f(\zeta)\ \mathrm{d}\zeta}{\zeta - t}$ between (1) and (3) gives

$$f(t) = \frac{1}{a(t)^2 - b(t)^2} \left(a(t)g(t) - \frac{b(t)}{\pi \mathrm{i}} \fint_\Gamma \frac{g(\zeta)\ \mathrm{d}\zeta}{\zeta - t} \right).$$

$\boxed{9}$ **Unseen. the last part will require some thought**

(d) [6 marks]

$$2f(t) + \frac{1}{\pi \mathrm{i}} \fint_\Gamma \left(\frac{1}{\zeta - t} + \alpha \right) f(\zeta)\ \mathrm{d}\zeta = \frac{1}{t}.$$

Writing

$$\frac{1}{\pi \mathrm{i}} \int_\Gamma f(\zeta)\ \mathrm{d}\zeta = \beta$$

gives

$$2f(t) + \frac{1}{\pi \mathrm{i}} \fint_\Gamma \frac{f(\zeta)\ \mathrm{d}\zeta}{\zeta - t}\ \mathrm{d}\zeta = \frac{1}{t} - \alpha\beta.$$

Since $a = 2$, $b = 1$, The solution from part (b) then gives

$$f(t) = \frac{1}{3}\left(\frac{2}{t} - 2\alpha\beta - \frac{1}{\pi \mathrm{i}}⨍_\Gamma \left(\frac{1}{\zeta} - \alpha\beta\right)\frac{1}{\zeta - t}\,\mathrm{d}\zeta\right).$$

using the results from part (b) gives

$$f(t) = \frac{1}{3}\left(\frac{2}{t} - 2\alpha\beta + \frac{1}{t} + \alpha\beta\right) = \frac{1}{t} - \frac{\alpha\beta}{3}.$$

To evaluate β we integrate to give

$$\beta = \frac{1}{\pi \mathrm{i}}\int_\Gamma \left(\frac{1}{\zeta} - \frac{\alpha\beta}{3}\right)\,\mathrm{d}\zeta = 2,$$

again by the Cauchy's Residue Theorem. Thus

$$f(t) = \frac{1}{t} - \frac{2\alpha}{3}.$$

$\boxed{6}$ **Unseen. Will require some thought**

3.

$$\int_0^\infty K(x-t)f(t)\,\mathrm{d}t = 3f(x) \qquad \text{for } x \geqslant 0,$$

where $K(x) = \mathrm{e}^{-|x|} + \mathrm{e}^{-2|x|}$.

(a) [5 marks] Define

$$f_+(x) = \begin{cases} 0 & x < 0, \\ f(x) & x > 0, \end{cases} \qquad h_-(x) = \begin{cases} \displaystyle\int_0^\infty K(x-t)f(t)\,\mathrm{d}t & x < 0, \\ 0 & x > 0. \end{cases}$$

Then

$$\int_{-\infty}^\infty K(x-t)f_+(t)\,\mathrm{d}t = 3f_+(t) + h_-(x) \quad \text{for } -\infty < x < \infty.$$

Since $f(x)$ is assumed to be $O(x)$, $\bar{f}_+(k)$ is holomorphic in $\mathrm{Im}(k) > 0$. Since, for $x < 0$,

$$h_-(x) = \mathrm{e}^x \int_0^\infty \mathrm{e}^{-t} f(t)\,\mathrm{d}t + \mathrm{e}^{2x}\int_0^\infty \mathrm{e}^{-2t} f(t)\,\mathrm{d}t = O\left(\mathrm{e}^x\right) \quad \text{as } x \to -\infty,$$

it follows that $\bar{h}_-(k)$ is holomorphic in $\mathrm{Im}(k) < 1$. Hence both $\bar{f}_+(k)$ and $\bar{h}_-(k)$ are holomorphic in the overlap strip $0 < \mathrm{Im}(k) < 1$.

$\boxed{5}$ **Bookwork**

(b) [12 marks] Taking the Fourier transform and using the Convolution Theorem gives

$$\bar{K}(k)\,\bar{f}_+(k) = 3\bar{f}_+(t) + \bar{h}_-(k). \tag{4}$$

The Fourier transform of K is

$$\begin{aligned} \bar{K}(k) &= \int_{-\infty}^0 \left(\mathrm{e}^{x+\mathrm{i}kx} + \mathrm{e}^{2x+\mathrm{i}kx}\right)\,\mathrm{d}x + \int_0^\infty \left(\mathrm{e}^{-x+\mathrm{i}kx} + \mathrm{e}^{-2x+\mathrm{i}kx}\right)\,\mathrm{d}x \\ &= \left[\frac{\mathrm{e}^{x+\mathrm{i}kx}}{1+\mathrm{i}k} + \frac{\mathrm{e}^{2x+\mathrm{i}kx}}{2+\mathrm{i}k}\right]_{-\infty}^0 + \left[\frac{\mathrm{e}^{-x+\mathrm{i}kx}}{-1+\mathrm{i}k} + \frac{\mathrm{e}^{-2x+\mathrm{i}kx}}{-2+\mathrm{i}k}\right]_0^\infty \\ &= \frac{1}{1+\mathrm{i}k} + \frac{1}{2+\mathrm{i}k} + \frac{1}{1-\mathrm{i}k} + \frac{1}{2-\mathrm{i}k} = \frac{2}{1+k^2} + \frac{4}{4+k^2} = \frac{6(2+k^2)}{(1+k^2)(4+k^2)}. \end{aligned}$$

Thus

$$\bar{K}-3 = \frac{6(2+k^2)-3(1+k^2)(4+k^2)}{(1+k^2)(4+k^2)} = \frac{12+6k^2-12-15k^2-3k^4}{(1+k^2)(4+k^2)} = -\frac{3k^2(3+k^2)}{(1+k^2)(4+k^2)}$$

Thus (4) is

$$-\frac{3k^2(3+k^2)}{(1+k^2)(4+k^2)}\,\bar{f}_+(k) = \bar{h}_-(k) \quad \text{for } 0 < \mathrm{Im}(k) < 1. \tag{5}$$

We factorise

$$-\frac{3k^2(3+k^2)}{(1+k^2)(4+k^2)} = \frac{K_+(k)}{K_-(k)}, \quad \text{where } K_+(k) = \frac{3k^2(k+\sqrt{3}\,\mathrm{i})}{(k+\mathrm{i})(k+2\mathrm{i})}, \quad K_-(k) = -\frac{(k-\mathrm{i})(k-2\mathrm{i})}{(k-\sqrt{3}\,\mathrm{i})},$$

so that $K_+(k)$ is holomorphic in $\mathrm{Im}(k) > -1$ and $K_-(k)$ is holomorphic in $\mathrm{Im}(k) < \sqrt{3}$. Hence,

$$\frac{3k^2(k+\sqrt{3}\mathrm{i})}{(k+\mathrm{i})(k+2\mathrm{i})}\bar{f}_+(k) = -\frac{(k-\mathrm{i})(k-2\mathrm{i})}{(k-\sqrt{3}\,\mathrm{i})}\bar{h}_-(k) = E(k) \text{ (say)} \quad \text{for } 0 < \mathrm{Im}(k) < 1, \tag{6}$$

with the left-hand side holomorphic in $\mathrm{Im}(k) > 0$ and the right-hand side holomorphic in $\mathrm{Im}(k) < 1$. Thus the right-hand side of (6) is the analytic continuation of the left-hand side of (6) into the lower half-plane, so together they define an entire function $E(k)$.

To pin down $E(k)$, we need to consider the behaviour as $k \to \infty$. We are given that $\bar{f}_+(k)$ is $O\left(k^{-1}\right)$ as $k \to \infty$ in $\mathrm{Im}(k) > 0$, and that $h_-(x) = O\left(k^{-1}\right)$ as $k \to \infty$ in $\mathrm{Im}(k) < 1$. Thus $E(k)$ is bounded at infinity and therefore $E(k) = C$, a constant, by Liouville.

[12] **Seen before with a single exponential in the Kernel**

(c) [8 marks] Thus

$$\bar{f}_+(k) = C\frac{(k+\mathrm{i})(k+2\mathrm{i})}{3k^2(k+\sqrt{3}\mathrm{i})}.$$

Inverting

$$f = \frac{C}{2\pi}\int_{-\infty}^{\infty} \frac{(k+\mathrm{i})(k+2\mathrm{i})}{3k^2(k+\sqrt{3}\mathrm{i})}\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k$$

where the inversion contour must lie in $\mathrm{Im}(k) > 0$. Deforming to $\mathrm{Im}(k) = -\infty$ we pick up residues from $k = 0$ and $k = -\sqrt{3}\mathrm{i}$. The residue at $k = -\sqrt{3}\mathrm{i}$ is

$$\frac{C}{2\pi}\frac{(1-\sqrt{3})(2-\sqrt{3})}{9}\mathrm{e}^{-\sqrt{3}\,x} = \frac{C}{2\pi}\frac{(5-3\sqrt{3})}{9}\mathrm{e}^{-\sqrt{3}\,x}$$

The residue at $k = 0$ is

$$\frac{C}{2\pi}\frac{1}{3}\left(\frac{2\mathrm{i}}{\sqrt{3}\mathrm{i}} + \frac{\mathrm{i}}{\sqrt{3}\mathrm{i}} + \frac{\mathrm{i}(2\mathrm{i})}{(\sqrt{3}\mathrm{i})(-\sqrt{3}\mathrm{i})} + \frac{(\mathrm{i})(2\mathrm{i})(-\mathrm{i}x)}{\sqrt{3}\mathrm{i}}\right) = \frac{C}{2\pi}\frac{1}{3}\left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} - \frac{2}{3} + \frac{2x}{\sqrt{3}}\right)$$
$$= \frac{C}{18\pi}\left(3\sqrt{3} - 2 + 2\sqrt{3}\,x\right).$$

Thus

$$f(x) = -\frac{\mathrm{i}C}{9}\left((5-3\sqrt{3})\mathrm{e}^{-\sqrt{3}\,x} + 3\sqrt{3} - 2 + 2\sqrt{3}\,x\right).$$

[8] **Standard technique, but will require some thought, especially the residue at $k = 0$**