A12207W1

SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part C: Paper C5.6
Honour School of Mathematical and Theoretical Physics Part C: Paper C5.6
Master of Science in Mathematical Sciences: Paper C5.6
Master of Science in Mathematical and Theoretical Physics: Paper C5.6

Applied Complex Variables

TRINITY TERM 2022

Tuesday 07 June, 14:30pm to 16:15pm

You may submit answers to as many questions as you wish but only the best two will count for the total mark. All questions are worth 25 marks.
Candidates may bring a summary sheet into this exam consisting of (both sides of) one sheet of A4 paper containing material prepared in accordance with the guidance given by the Mathematical Institute.

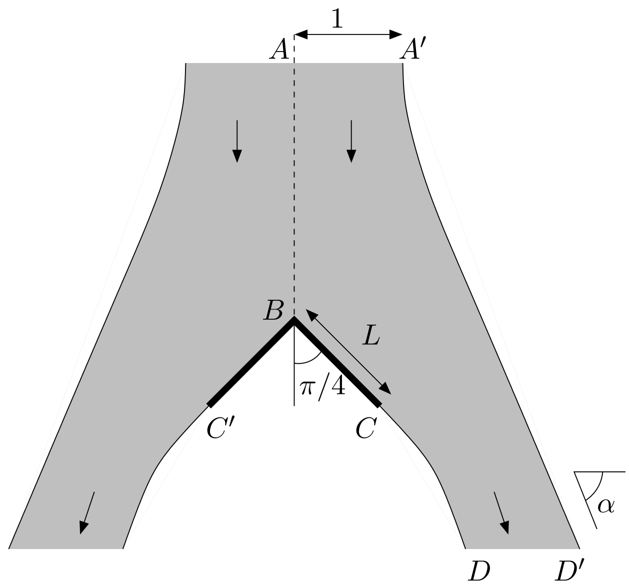
You should ensure that you observe the following points:

- start a new answer booklet for each question which you attempt.
- indicate on the front page of the answer booklet which question you have attempted in that booklet.
- cross out all rough working and any working you do not want to be marked. If you have used separate answer booklets for rough work please cross through the front of each such booklet and attach these answer booklets at the back of your work.
- hand in your answers in numerical order.

If you do not attempt any questions, you should still hand in an answer booklet with the front sheet completed.

Do not turn this page until you are told that you may do so.

1. A steady vertical two-dimensional inviscid incompressible fluid jet, of width 2 and speed 1 impinges on a V-shaped plate $C'BC$ as shown in the diagram. The plate has half-angle $\pi/4$ and side length L, and divides the jet symmetrically about the vertical line AB. The downstream right-hand jet DD' makes a downward angle α with the horizontal, with $\pi/4 < \alpha < \pi/2$. The points AA' are at $y = \infty$ and DD' at $y = -\infty$. The flow leaves the point C tangentially. Without loss of generality, take the streamfunction ψ and the potential ϕ to be zero at B.



(a) [8 marks] Show that the images of the right-hand half of the shaded fluid domain (i.e. that bounded by $ABCDD'A'$) in the potential w-plane and the hodograph w'-plane are a strip and a circular wedge respectively; sketch these regions indicating clearly the image of each of the labelled points.

(b) [9 marks] Show that

$$\zeta = \left(\frac{1 - (w')^4}{1 + (w')^4} \right)^2$$

maps the image of the right-hand half of the fluid domain in the w'-plane to the upper half-plane, with the point D mapped to $\zeta_D = -\tan^2 2\alpha$. Indicate clearly the positions in the ζ-plane of the points A, B and C.

(c) [8 marks] Show that w satisfies the equation

$$\mathrm{e}^{\pi w} = \frac{(1 - (w')^4)^2}{1 - 2(w')^4 \cos 4\alpha + (w')^8}.$$

Deduce that

$$\phi = \frac{2}{\pi} \log \sec 2\alpha$$

at the point C.

2. (a) [5 marks] Let Γ be a directed smooth contour in the complex plane and

$$w(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z},$$

where f is continuous on Γ and holomorphic in a neighbourhood of the point $t \in \Gamma$. Show that the limiting values of $w(z)$ as Γ is approached from either side are $w_\pm(t)$, where

$$w_\pm(t) = \pm\frac{f(t)}{2} + \frac{1}{2\pi \mathrm{i}} ⨍_\Gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - t},$$

and you should define the integral $⨍$ precisely; indicate with a sketch which side of the contour is $+$ and which is $-$.

(b) [13 marks] Now suppose that Γ is closed and that f is differentiable on Γ. Show that $w(z) \to 0$ as $z \to \infty$.

By defining $W(z) = w(z)$ for z inside Γ, and $W(z) = -w(z)$ for z outside Γ, show that if

$$f'(t) + ⨍_\Gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - t} = 1 \qquad (\dagger)$$

then

$$w'(z) + \pi \mathrm{i}\, W(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{\mathrm{d}\zeta}{\zeta - z}.$$

Hence find w and deduce that the general solution to $(\dagger)$ is

$$f(z) = \frac{1}{\pi \mathrm{i}} + C\mathrm{e}^{-\pi \mathrm{i} z},$$

where C is an arbitrary constant.

(c) [7 marks] Find the general solution to

$$f'(t) + ⨍_\Gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - t} = g(t)$$

where $g(z)$ is holomorphic on and inside Γ, and your answer will involve an integral of g.

3. Consider the integral equation

$$f(x) + \int_0^\infty K(x-t) f(t) \ \mathrm{d}t = \mathrm{e}^{-2x} \qquad \text{for } x \geqslant 0,$$

where

$$K(x) = \begin{cases} \mathrm{e}^x & x < 0, \\ 1 & x \geqslant 0, \end{cases}$$

and you may assume that $f : [0, \infty) \to \mathbb{R}$ is continuous and bounded.

(a) [6 marks] Show how to convert this to the integral equation

$$f_+(x) + \int_{-\infty}^\infty K(x-t) f_+(t) \ \mathrm{d}t = h_-(x) + g_+(x)$$

on the full range $-\infty < x < \infty$, and explain how $f_+(x)$, $g_+(x)$ and $h_-(x)$ are defined. State for which regions of the complex k-plane the Fourier transforms $\bar{f}_+(k)$, $\bar{g}_+(k)$ and $\bar{h}_-(k)$ are defined and holomorphic.

(b) [12 marks] Show that

$$\frac{(k^2 - \mathrm{i}k + 1)}{k(k - \mathrm{i})} \bar{f}_+(k) = \bar{h}_-(k) + \frac{1}{2 - \mathrm{i}k}.$$

Deduce that

$$\frac{(k - \mathrm{i}\lambda_-)}{k} \bar{f}_+(k) - \frac{3\mathrm{i}}{(2 + \lambda_+)(k + 2\mathrm{i})} = \frac{(k - \mathrm{i})}{(k - \mathrm{i}\lambda_+)} \bar{h}_-(k) + \frac{\mathrm{i}(\lambda_+ - 1)}{(\lambda_+ + 2)(k - \mathrm{i}\lambda_+)}, \qquad (\ddagger)$$

in a strip $\alpha < \mathrm{Im}(k) < \beta$, and give suitable values of α and β, where

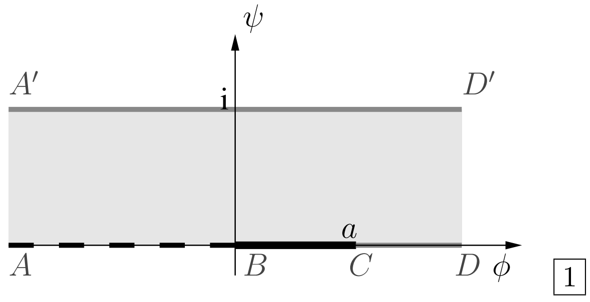
$$\lambda_\pm = \frac{1 \pm \sqrt{5}}{2}.$$

Explain clearly why the left- and right-hand sides of ($\ddagger$) must be zero.
[*You may assume without proof that $\bar{f}_+(k)$ and $\bar{h}_-(k)$ are $O(k^{-1})$ as $k \to \infty$.*]

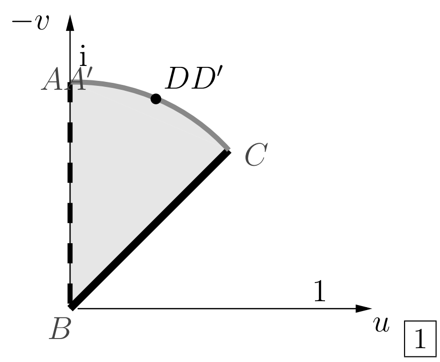
(c) [7 marks] Hence solve for $\bar{f}_+(k)$ and, by inverting the Fourier transform, show that

$$f(x) = \frac{6\mathrm{e}^{-2x}}{5} - \frac{3}{10}(\sqrt{5} - 1)\mathrm{e}^{-(\sqrt{5}-1)x/2}.$$

1. (a) [8 marks] Since $\psi = 0$ on $ABCA'$ and $\psi_x = -\phi_y = 1$ at infinity the stream function takes the value 1 on $A'D'$ [1]. Thus the potential plane is the strip $0 < \psi < 1$, $-\infty < \phi < \infty$, where $\phi(C) = a > 0$ is unknown [1].



The velocity at A and A' is $(0, -1)$ [1]. Since $|w'|$ is constant on $A'D'$ we must have $|w'| = 1$ on $A'D'$, and therefore also on CD. The velocity at B is 0 [1]. The velocity at C is of unit modulus making an angle $-\pi/4$ with the x-axis so $w' = e^{i\pi/4}$ [1]. The velocity at DD' is of unit modulus making an angle $-\alpha$ with the x-axis, so $w' = e^{i\alpha}$ [1]. On AB we know $u = 0$. Finally on BC we know (u, v) makes an angle $-\pi/4$ with the x-axis.

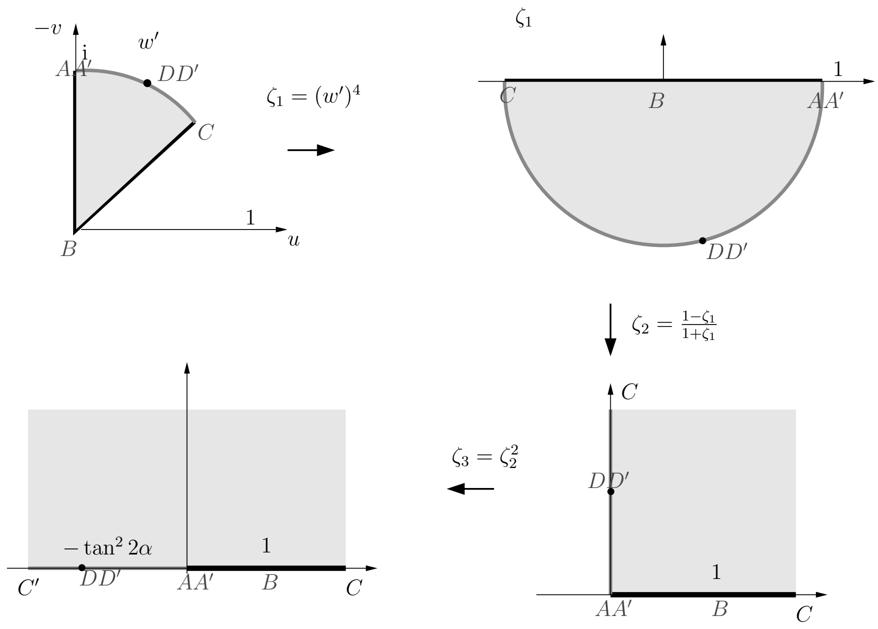


□ **New example, but familiar method. BC will take some thought.**

(b) [9 marks] For the hodograph plane, first get rid of the corner at the origin by raising to the power 4: $\zeta_1 = (w')^4$ [1]. This gets us the the lower semi-circle [1]. Then use the mobius map

$$\zeta_2 = \frac{1 - \zeta_1}{\zeta_1 + 1}$$

which sends 1 to 0, -1 to ∞, and 0 to 1 [1]. Thus this maps the semicircle to the first quadrant [1]. Now square to get the upper half plane [1].



Thus

$$\zeta_3 = \left(\frac{1-(w')^4}{1+(w')^4}\right)^2,$$

and DD' is the point

$$\zeta_D = \left(\frac{1-e^{4i\alpha}}{1+e^{4i\alpha}}\right)^2 = \left(\frac{e^{2i\alpha}-e^{-2i\alpha}}{e^{2i\alpha}+e^{2i\alpha}}\right)^2 = -\tan^2 2\alpha. \quad \boxed{1}$$

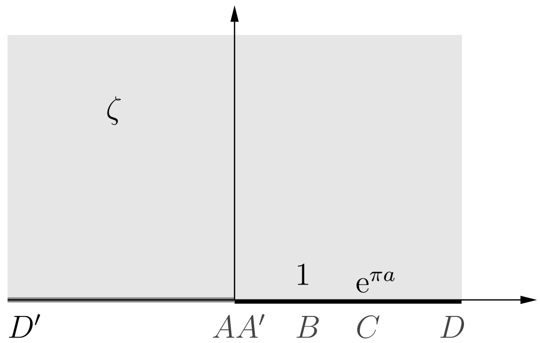
$\zeta_A = 0 \quad \boxed{1}, \quad \zeta_B = 1, \quad \boxed{1}, \quad \zeta_C = \infty, \quad \boxed{1}.$

□ **New example, but some of the maps have been seen before. Will require some thought to calculate ζ_D.**

(c) [8 marks] To map the strip in the potential plane to the upper-half plane, we use

$$\zeta = e^{\pi w}. \quad \boxed{1}$$

This maps B to 1, AA' to 0 and DD' to ∞. $\boxed{1}$



In the hodograph ζ_3-plane the points are in the wrong places. We need to send 0 to 0, 1 to 1 and $-\tan^2 2\alpha$ to infinity $\boxed{1}$. Thus we need to set

$$\begin{aligned}
\zeta &= \frac{\zeta_3(1+\tan^2 2\alpha)}{\zeta_3+\tan^2 2\alpha} = \frac{\zeta_3}{\zeta_3\cos^2 2\alpha + \sin^2 2\alpha} = \frac{(1-(w')^4)^2}{(1-(w')^4)^2\cos^2 2\alpha + (1+(w')^4)^2\sin^2 2\alpha} \\
&= \frac{(1-(w')^4)^2}{1+(w')^8+2(w')^4(\sin^2 2\alpha - \cos^2 2\alpha)} = \frac{(1-(w')^4)^2}{1+(w')^8-2(w')^4\cos 4\alpha}. \quad \boxed{3}
\end{aligned}$$

Thus w satisfies the differential equation

$$\mathrm{e}^{\pi w} = \frac{(1-(w')^4)^2}{1-2(w')^4\cos 4\alpha + (w')^8}.$$

The point C is $\zeta_3 = \infty$ $\boxed{1}$, so that

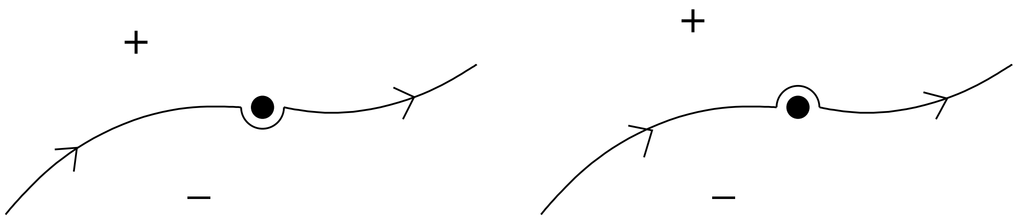
$$\zeta = 1 + \tan^2 2\alpha = \frac{1}{\cos^2 2\alpha} = \mathrm{e}^{\pi a}. \quad \boxed{1}$$

Thus

$$a = \frac{1}{\pi}\log(1/\cos^2 2\alpha) = \frac{2}{\pi}\log\sec 2\alpha.$$

□ **New example. Need to be careful to make sure all the points are in the right place. Will take some thought to identify a**

2. (a) [5 marks] Label the left-hand side of Γ as "+" and the right-hand side as "-" $\boxed{1}$. As $z \to t \in \Gamma$ from the plus side indent the contour with a small semi-circle around t as shown in (a) $\boxed{1}$.



(a) (b)

Then

$$w_+(t) = \lim_{\epsilon\to 0}\frac{1}{2\pi\mathrm{i}}\left(\int_{\text{one end}}^{t-\epsilon}\frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - t} + \int_{\text{semicircle}|\zeta-t|=\epsilon}\frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - t} + \int_{\text{other end}}^{t-\epsilon}\frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - t}\right). \quad \boxed{1}$$

As $\epsilon \to 0$ the semicircle gives a contribution $\dfrac{1}{2\pi\mathrm{i}} \times \pi\mathrm{i}f(t)$ $\boxed{1}$. Thus

$$w_+(t) = \frac{f(t)}{2} + \frac{1}{2\pi\mathrm{i}}⨍_\Gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - t},$$

where

$$⨍ = \lim_{\epsilon\to 0}\left(\int^{t-\epsilon} + \int_{t+\epsilon}\right). \quad \boxed{1}$$

As $z \to t \in \Gamma$ from the minus side we need to indent the contour on the other side with a small semi-circle around t as shown in (b). The semicircle now gives a contribution $-\dfrac{1}{2\pi\mathrm{i}} \times \pi\mathrm{i}f(t)$ as $\epsilon \to 0$, so that

$$w_-(t) = -\frac{f(t)}{2} + \frac{1}{2\pi\mathrm{i}}⨍_\Gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - t}.$$

□ **Bookwork**

(b) [13 marks] With

$$w(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f(\zeta)\, \mathrm{d}\zeta}{\zeta - z}, \qquad w \sim -\frac{1}{z}\frac{1}{2\pi \mathrm{i}} \int_\Gamma f(\zeta)\, \mathrm{d}\zeta \to 0 \text{ as } z \to \infty$$

since $\int_\Gamma f(\zeta)\, \mathrm{d}\zeta$ exists (Γ is bounded and f is differentiable on Γ). $\boxed{1}$
Define $W(z)$ such that $W(z) = w(z)$ for z inside Γ and $W(z) = -w(z)$ for z outside Γ. Then we look for a Cauchy integral representation of $G = w' + \pi \mathrm{i}\, W$ in the form

$$G(z) = w'(z) + \pi \mathrm{i}\, W(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{h(\zeta)\, \mathrm{d}\zeta}{\zeta - z}, \quad \boxed{2}$$

where we need to find $h(\zeta)$. The Plemelj formulae now give

$$G_\pm(t) = \pm\frac{1}{2}h(t) + \frac{1}{2\pi \mathrm{i}} ⨍_\Gamma \frac{h(\zeta)\, \mathrm{d}\zeta}{\zeta - t}.$$

Thus

$$\begin{aligned} h(t) &= G_+(t) - G_-(t) = w'_+(t) - w'_-(t) + \pi \mathrm{i} W_+ - \pi \mathrm{i} W_- \\ &= w'_+(t) - w'_-(t) + \pi \mathrm{i}(w_+ + w_-) = f'(t) + ⨍_\Gamma \frac{f(\zeta)\, \mathrm{d}\zeta}{\zeta - t} = 1, \quad \boxed{2} \end{aligned}$$

from the given equation for f, where we have used the fact that

$$w'_+(t) - w'_-(t) = f'(t) \quad \boxed{1}$$

which follows either from differentiating $w_+ - w_- = f$ along Γ, or from observing

$$w'(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f(\zeta)\, \mathrm{d}\zeta}{(\zeta - z)^2} = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f'(\zeta)\, \mathrm{d}\zeta}{\zeta - z},$$

on integrating by parts (since Γ is closed). Thus

$$w'(z) + \pi \mathrm{i}\, W(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{\mathrm{d}\zeta}{\zeta - z}. \quad \boxed{1}$$

Thus, for z inside Γ,

$$w(z) = \frac{1}{\pi \mathrm{i}} - \frac{1}{\pi \mathrm{i}} w'(z), \quad \boxed{1}$$

while for z outside Γ,

$$-w(z) = -\frac{1}{\pi \mathrm{i}} w'(z). \quad \boxed{1}$$

Thus

$$w(z) = \begin{cases} \dfrac{1}{\pi \mathrm{i}} + C_+ \mathrm{e}^{-\pi \mathrm{i} z} & z \text{ inside } \Gamma, \\ C_- \mathrm{e}^{\pi \mathrm{i} z} & z \text{ outside } \Gamma, \end{cases} \quad \boxed{2}$$

But $w \to 0$ as $z \to \infty$ so that $C_- = 0$ $\boxed{1}$. Then

$$f(t) = w_+(t) = \frac{1}{\pi \mathrm{i}} + C_+ \mathrm{e}^{-\pi \mathrm{i} t}. \quad \boxed{1}$$

□ **Unseen. Will require some thought**

(c) [7 marks] Proceeding as before by seeking a Cauchy integral representation for $G = w' + \pi \mathrm{i} W$ in the form

$$G(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{h(\zeta)\ \mathrm{d}\zeta}{\zeta - z},$$

we have

$$\begin{aligned} h(t) &= G_+(t) - G_-(t) = w'_+(t) - w'_-(t) + \pi \mathrm{i} W_+ - \pi \mathrm{i} W_- \\ &= w'_+(t) - w'_-(t) + \pi \mathrm{i}(w_+ + w_-) = f'(t) + \int_\Gamma \frac{f(\zeta)\ \mathrm{d}\zeta}{\zeta - t} = g(t), \quad \boxed{1} \end{aligned}$$

so that

$$w'(z) + \pi \mathrm{i} W(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{g(\zeta)\,\mathrm{d}\zeta}{\zeta - z}. \quad \boxed{1}$$

Thus, for z inside Γ,

$$w(z) = \frac{g(z)}{\pi \mathrm{i}} - \frac{1}{\pi \mathrm{i}} w'(z), \quad \boxed{1}$$

while for z outside Γ,

$$-w(z) = -\frac{1}{\pi \mathrm{i}} w'(z), \quad \boxed{1}$$

since $g(z)$ is holomorphic inside Γ. Thus

$$w(z) = \begin{cases} \mathrm{e}^{-\pi \mathrm{i} z} \displaystyle\int_{z_0}^{z} g(\zeta) \mathrm{e}^{\mathrm{i}\pi\zeta}\ \mathrm{d}\zeta + C_+ \mathrm{e}^{-\pi \mathrm{i} z} & z \text{ inside } \Gamma, \\ C_- \mathrm{e}^{\pi \mathrm{i} z} & z \text{ outside } \Gamma, \end{cases} \quad \boxed{2}$$

where z_0 is any point inside Γ. But $w \to 0$ as $z \to \infty$ so that $C_- = 0$. Then

$$f(t) = w_+(t) = \mathrm{e}^{-\pi \mathrm{i} t} \int_{z_0}^{t} g(\zeta) \mathrm{e}^{\mathrm{i}\pi\zeta}\ \mathrm{d}\zeta + C_+ \mathrm{e}^{-\pi \mathrm{i} t}. \quad \boxed{1}$$

□ **Unseen. Will require some thought**

3.

$$f(x) + \int_0^\infty K(x-t) f(t)\ \mathrm{d}t = \mathrm{e}^{-2x} \qquad \text{for } x \geqslant 0,$$

where

$$K(x) = \begin{cases} \mathrm{e}^x & x < 0, \\ 1 & x \geqslant 0. \end{cases}$$

(a) [6 marks] Define

$$f_+(x) = \begin{cases} 0 & x < 0, \\ f(x) & x \geqslant 0, \end{cases} \qquad h_-(x) = \begin{cases} \displaystyle\int_0^\infty K(x-t) f(t)\,\mathrm{d}t & x < 0, \\ 0 & x \geqslant 0. \end{cases}$$

$$g_+(x) = \begin{cases} 0 & x < 0, \\ \mathrm{e}^{-2x} & x \geqslant 0. \end{cases} \quad \boxed{3}$$

Then

$$f_+(x) + \int_{-\infty}^{\infty} K(x-t) f_+(t)\,\mathrm{d}t = h_-(x) + g_+(x) \quad \text{for } -\infty < x < \infty,$$

Since $f(x)$ is assumed to be bounded, $\bar{f}_+(k)$ is holomorphic in $\mathrm{Im}(k) > 0$ $\boxed{1}$. Since, for $x < 0$,

$$h_-(x) = \mathrm{e}^x \int_0^\infty \mathrm{e}^{-t} f(t)\,\mathrm{d}t = O\left(\mathrm{e}^x\right) \quad \text{as } x \to -\infty,$$

it follows that $\bar{h}_-(k)$ is holomorphic in $\mathrm{Im}(k) < 1$ $\boxed{1}$. Finally

$$\bar{g}_+(k) = \int_0^\infty \mathrm{e}^{-2x+\mathrm{i}kx}\,\mathrm{d}x = \left[\frac{\mathrm{e}^{-2x+\mathrm{i}kx}}{\mathrm{i}k-2}\right]_0^\infty = \frac{1}{2-\mathrm{i}k}$$

providing $\mathrm{Im}(k) > -2$ $\boxed{1}$.

$\square$ **Mainly bookwork, though $\bar{h}_-$ takes a little thought.**

(b) [12 marks] Taking the Fourier transform and using the Convolution Theorem gives

$$\bar{f}_+(k) + \bar{K}(k)\,\bar{f}_+(k) = \bar{h}_-(k) + \bar{g}_+(k) \quad \boxed{1}. \tag{1}$$

The Fourier transform of K is

$$\begin{aligned}\bar{K}(k) &= \int_{-\infty}^0 \mathrm{e}^{x+\mathrm{i}kx}\,\mathrm{d}x + \int_0^\infty \mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x \\ &= \left[\frac{\mathrm{e}^{x+\mathrm{i}kx}}{1+\mathrm{i}k}\right]_{-\infty}^0 + \left[\frac{\mathrm{e}^{\mathrm{i}kx}}{\mathrm{i}k}\right]_0^\infty = \frac{1}{1+\mathrm{i}k} - \frac{1}{\mathrm{i}k} = \frac{1}{k(k-\mathrm{i})}, \quad \boxed{1}\end{aligned}$$

defined for $0 < \mathrm{Im}(k) < 1$ $\boxed{1}$.
Thus (1) is

$$\frac{(k^2 - \mathrm{i}k + 1)}{k(k-\mathrm{i})}\,\bar{f}_+(k) = \bar{h}_-(k) + \frac{1}{2-\mathrm{i}k} \quad \text{for } 0 < \mathrm{Im}(k) < 1. \quad \boxed{1} \tag{2}$$

$\square$ **Similar to worked example**

We first factorise

$$1 + \bar{K} = \frac{k^2 - \mathrm{i}k + 1}{k(k-\mathrm{i})} = \frac{(k-\lambda_+\mathrm{i})(k-\lambda_-\mathrm{i})}{k(k-\mathrm{i})} \text{ where } \lambda_\pm = \frac{1 \pm \sqrt{5}}{2}. \quad \boxed{1}$$

Thus

$$\frac{(k-\lambda_+\mathrm{i})(k-\lambda_-\mathrm{i})}{k(k-\mathrm{i})} = \frac{K_+(k)}{K_-(k)}, \quad \text{where } K_+(k) = \frac{(k-\lambda_-\mathrm{i})}{k}, \quad K_-(k) = \frac{(k-\mathrm{i})}{(k-\lambda_+\mathrm{i})}, \quad \boxed{1}$$

so that $K_+(k)$ is holomorphic in $\mathrm{Im}(k) > 0$ and $K_-(k)$ is holomorphic in $\mathrm{Im}(k) < \lambda_+$. Then

$$\frac{(k-\lambda_-\mathrm{i})}{k}\bar{f}_+(k) = \frac{(k-\mathrm{i})}{(k-\lambda_+\mathrm{i})}\bar{h}_-(k) + \frac{\mathrm{i}}{k+2\mathrm{i}}\frac{(k-\mathrm{i})}{(k-\lambda_+\mathrm{i})} \quad \text{for } 0 < \mathrm{Im}(k) < 1. \quad \boxed{1} \tag{3}$$

We now need an additive factorisation

$$\begin{aligned}\frac{\mathrm{i}}{(k+2\mathrm{i})}\frac{(k-\mathrm{i})}{(k-\lambda_+\mathrm{i})} &= \frac{\mathrm{i}}{(k+2\mathrm{i})}\frac{(-2\mathrm{i}-\mathrm{i})}{(-2\mathrm{i}-\lambda_+\mathrm{i})} + \frac{\mathrm{i}}{(\lambda_+\mathrm{i}+2\mathrm{i})}\frac{(\lambda_+\mathrm{i}-\mathrm{i})}{(k-\lambda_+\mathrm{i})} \\ &= \frac{3\mathrm{i}}{(2+\lambda_+)(k+2\mathrm{i})} + \frac{\mathrm{i}(\lambda_+-1)}{(\lambda_++2)(k-\lambda_+\mathrm{i})} \quad \boxed{1}\end{aligned}$$

with the first term is holomorphic in $\mathrm{Im}(k) > -2$ and the second term is holomorphic in $\mathrm{Im}(k) < \lambda_+$. In (3) this gives

$$\frac{(k-\lambda_-\mathrm{i})}{k}\bar{f}_+(k) - \frac{3\mathrm{i}}{(2+\lambda_+)(k+2\mathrm{i})} = \frac{(k-\mathrm{i})}{(k-\lambda_+\mathrm{i})}\bar{h}_-(k) + \frac{\mathrm{i}(\lambda_+-1)}{(\lambda_++2)(k-\lambda_+\mathrm{i})} = E(k), \quad \boxed{1} \tag{4}$$

say, for $0 < \mathrm{Im}(k) < 1$. $\boxed{1}$ The left-hand side is holomorphic in $\mathrm{Im}(k) > 0$, while the right-hand side is holomorphic in $\mathrm{Im}(k) < 1$. Thus the right-hand side of (4) is the analytic continuation of the left-hand side of (4) into the lower half-plane, so together they define an entire function $E(k)$. $\boxed{1}$

To pin down $E(k)$, we need to consider the behaviour as $k \to \infty$. We are given that $\bar{f}_+(k)$ is $O\left(k^{-1}\right)$ as $k \to \infty$ in $\mathrm{Im}(k) > 0$, and that $h_-(x) = O\left(k^{-1}\right)$ as $k \to \infty$ in $\mathrm{Im}(k) < 1$. Thus $E(k)$ is tends to zero at infinity and therefore $E(k) \equiv 0$ by Liouville. $\boxed{1}$

□ **Similar equations seen without the inhomogeneous term g, so the extra additive decomposition is new, although it has been seen in other contexts.**

(c) [7 marks] Thus

$$\bar{f}_+(k) = \frac{3\mathrm{i}}{(2+\lambda_+)(k+2\mathrm{i})}\frac{k}{(k-\lambda_-\mathrm{i})}. \quad \boxed{1}$$

Inverting

$$f = \frac{3\mathrm{i}}{2\pi(2+\lambda_+)}\int_{-\infty}^{\infty}\frac{k}{(k+2\mathrm{i})(k-\lambda_-\mathrm{i})}\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k \quad \boxed{1}$$

where the inversion contour must lie in $\mathrm{Im}(k) > 0$ $\boxed{1}$. Deforming to $\mathrm{Im}(k) = -\infty$ we pick up residues from $k = -2\mathrm{i}$ and $k = \lambda_-\mathrm{i}$ $\boxed{1}$. The residue at $k = -2\mathrm{i}$ is

$$\frac{3\mathrm{i}}{2\pi(2+\lambda_+)}\frac{2}{(2+\lambda_-)}\mathrm{e}^{-2x} = \frac{6\mathrm{i}}{2\pi}\frac{4}{(5+\sqrt{5})(5-\sqrt{5})}\mathrm{e}^{-2x} = \frac{6\mathrm{i}}{10\pi}\mathrm{e}^{-2x} \quad \boxed{1}$$

The residue at $k = \lambda_-\mathrm{i}$ is

$$\frac{3\mathrm{i}}{2\pi(2+\lambda_+)}\frac{\lambda_-}{(\lambda_-+2)}\mathrm{e}^{\lambda_-x} = \frac{3\lambda_-\mathrm{i}}{2\pi}\frac{4}{(5+\sqrt{5})(5-\sqrt{5})}\mathrm{e}^{\lambda_-x} = \frac{3\lambda_-\mathrm{i}}{10\pi}\mathrm{e}^{\lambda_-x} \quad \boxed{1}.$$

Thus (minus sign because of clockwise integral) $\boxed{1}$

$$f(x) = 30\mathrm{e}^{-2x} + \frac{3}{5}\lambda_-\mathrm{e}^{\lambda_-x} = \frac{6\mathrm{e}^{-2x}}{5} - \frac{3}{10}(\sqrt{5}-1)\mathrm{e}^{-(\sqrt{5}-1)x/2}.$$

□ **Standard technique, but new example, and will require some thought**