A12207W1

SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part C: Paper C5.6
Honour School of Mathematics and Statistics Part C: Paper C5.6
Honour School of Mathematical and Theoretical Physics Part C: Paper C5.6
Master of Science in Mathematical Sciences: Paper C5.6
Master of Science in Mathematical and Theoretical Physics: Paper C5.6

Applied Complex Variables

TRINITY TERM 2023

Tuesday 06 June, 9:30am to 11:15am

You may submit answers to as many questions as you wish but only the best two will count for the total mark. All questions are worth 25 marks.

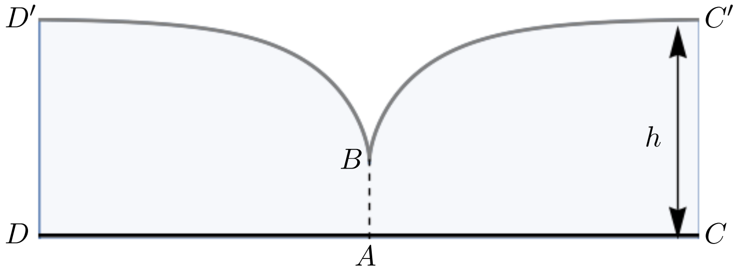
You should ensure that you observe the following points:

- start a new answer booklet for each question which you attempt.
- indicate on the front page of the answer booklet which question you have attempted in that booklet.
- cross out all rough working and any working you do not want to be marked. If you have used separate answer booklets for rough work please cross through the front of each such booklet and attach these answer booklets at the back of your work.
- hand in your answers in numerical order.

If you do not attempt any questions, you should still hand in an answer booklet with the front sheet completed.

Do not turn this page until you are told that you may do so

1. A two-dimensional, inviscid, irrotational, incompressible fluid lies above a flat plate DC. The fluid is of uniform depth h at infinity, with a free surface $D'BC'$ which is drawn into a cusp at the point B due to a steady sink of strength Q which lies on the flat plate at the point A. Without loss of generality, take the streamfunction ψ and the potential ϕ to be zero at B.



(a) [3 marks] Show that on AC the streamfunction takes the value $Q/4$. Give the dimensional scalings of position, velocity and potential such that the nondimensionalised model has fluid velocity $(-1, 0)$ and depth 1 as $x \to \infty$. Use the nondimensional model henceforth.

(b) [6 marks] Show that the images of the right-hand half of the shaded fluid domain (i.e. that bounded by $ABC'CA$) in the potential w-plane and the hodograph w'-plane are a strip and a quarter-plane minus a quarter-circle respectively; sketch these regions indicating clearly the image of each of the labelled points.

(c) [6 marks] Show that

$$\zeta = -\left(\frac{1-(w')^2}{1+(w')^2}\right)^2$$

maps the image of the right-hand half of the fluid domain in the w'-plane to the upper half-plane. Indicate clearly the positions in the ζ-plane of the points A, B and C.

(d) [5 marks] Show that w satisfies the equation

$$\mathrm{e}^{\pi w} = -\frac{(1-(w')^2)^2}{4(w')^2}.$$

(e) [5 marks] Show that the free surface is given by

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{2\cos^2\theta}{\pi\sin\theta}, \qquad \frac{\mathrm{d}y}{\mathrm{d}\theta} = \frac{2\cos\theta}{\pi},$$

where θ is the angle the tangent (in the flow direction) makes with the x-axis. What is the position of the point B? How would the picture change if A were a point source instead of a point sink?

2. (a) [5 marks] Let Γ be a directed smooth contour in the complex plane and set

$$w(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z},$$

where f is continuous on Γ and holomorphic in a neighbourhood of the point $t \in \Gamma$. Show that the limiting values of $w(z)$ as Γ is approached from either side are $w_\pm(t)$, where

$$w_\pm(t) = \pm\frac{f(t)}{2} + \frac{1}{2\pi \mathrm{i}} ⨍_\Gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - t},$$

and you should define the integral $⨍$ precisely; indicate with a sketch which side of the contour is $+$ and which is $-$.

(b) [5 marks] The complex velocity describing flow past a thin aerofoil is represented by a function $w(z)$ which is holomorphic on $\mathbb{C}\backslash[0,1]$ and has limiting values $w_\pm$ on $y = 0\pm$, $0 < x < 1$, that satisfy

$$w_+(x) + w_-(x) = 2\mathrm{i}g'(x),$$

where $g(x)$ is a smooth real-valued function describing the shape of the aerofoil. By introducing a suitable auxiliary function $\tilde{w}(z)$, show that a solution for $w(z)$ can be written in the form

$$w(z) = \frac{\tilde{w}(z)}{\pi} \int_0^1 \frac{g'(\xi)}{\tilde{w}_+(\xi)(\xi - z)}\,\mathrm{d}\xi + \tilde{w}(z)H(z),$$

where $H(z)$ is holomorphic on $\mathbb{C}\backslash\{0,1\}$.

(c) [5 marks] Suppose that, in addition $w = O(z^{-1/2})$ as $z \to 0$, $w = O(1)$ as $z \to 1$, and $w = O(1/z)$ as $z \to \infty$. Show that

$$w(z) = \left(\frac{1-z}{z}\right)^{1/2} \frac{1}{\pi} \int_0^1 \frac{\xi^{1/2} g'(\xi)}{(1-\xi)^{1/2}(\xi - z)}\,\mathrm{d}\xi.$$

Define carefully which branch of the square root is used.

(d) [5 marks] By considering a contour that encircles the branch cut and deforming to a large circle show that when $g(x) = \log(1 + x^2)$,

$$w(z) = \frac{2^{3/4}(1-z)^{1/2}(\sin(\pi/8) + \sin(3\pi/8)z)}{z^{1/2}(z^2+1)} + \frac{2\mathrm{i}z}{(1+z^2)}.$$

(e) [5 marks] Find a bounded solution $f(x)$ to the singular integral equation

$$\frac{1}{\pi} ⨍_0^1 \frac{f(\xi)}{\xi - x}\,\mathrm{d}\xi + \frac{4x}{1+x^2} = 0.$$

3. (a) [3 marks] Let

$$f_+(x) = \begin{cases} x & x > 0, \\ 0 & x < 0. \end{cases}$$

For what values of $k \in \mathbb{C}$ is the Fourier transform $\bar{f}_+(k)$ defined? Evaluate $\bar{f}_+(k)$. To what region of the complex k-plane can it be analytically continued?

(b) [5 marks] The function $G(k)$ is holomorphic in a strip $\Omega = \{k \in \mathbb{C} : \alpha < \mathrm{Im}(k) < \beta\}$ and satisfies $G(k) \to 0$ as $k \to \infty$ in Ω. Show that $G(k)$ may be decomposed as

$$G(k) = G_+(k) - G_-(k),$$

where $G_+(k)$ is holomorphic in $\mathrm{Im}(k) > \gamma_+$ and $G_-(k)$ is holomorphic in $\mathrm{Im}(k) < \gamma_-$, for $\alpha < \gamma_+ < \gamma_- < \beta$. Give explicit formulae for $G_+(k)$ and $G_-(k)$, in terms of integrals along specified contours in the complex k-plane.

(c) [7 marks] Clearly define branches of the multifunctions $(k - \mathrm{i})^{1/2}$ and $(k + \mathrm{i})^{1/2}$ which are holomorphic and have positive real part in the strip $-1 < \mathrm{Im}(k) < 1$. By evaluating the integrals in part (b), or otherwise, express the function

$$G(k) = \frac{(k - \mathrm{i})^{1/2}}{k^2}$$

as $G_+(k) - G_-(k)$ where $G_+(k)$ is holomorphic in $\mathrm{Im}(k) > 0$ and $G_-(k)$ is holomorphic in $\mathrm{Im}(k) < 1$.

(d) [5 marks] The function $u(x, y)$ satisfies the partial differential equation

$$\nabla^2 u = u \quad \text{for } y > 0$$

subject to

$$\frac{\partial u}{\partial y}(x, 0) = 0 \quad \text{for } x < 0, \qquad u(x, 0) = x \quad \text{for } x \geqslant 0,$$

with $u \to 0$ as $y \to \infty$. Define

$$f_-(x) = \begin{cases} u(x, 0) & x < 0, \\ 0 & x \geqslant 0, \end{cases} \qquad g_+(x) = \begin{cases} 0 & x < 0, \\ \partial u/\partial y(x, 0) & x \geqslant 0. \end{cases}$$

Assuming that the Fourier transforms $\bar{g}_+(k)$ and $\bar{f}_-(k)$ are holomorphic in $\mathrm{Im}(k) > 0$ and $\mathrm{Im}(k) < 1$ respectively, show that

$$\bar{g}_+ + (k^2 + 1)^{1/2} \bar{f}_- = \frac{(k^2 + 1)^{1/2}}{k^2},$$

where you should be careful to define the branch of $(k^2 + 1)^{1/2}$.

(e) [5 marks] Assuming that both $k^{-1/2} \bar{g}_+(k)$ and $k^{1/2} \bar{f}_-(k)$ tend to zero as $k \to \infty$ deduce that

$$\bar{g}_+ = \frac{\mathrm{e}^{\mathrm{i}\pi/4}(k - 2\mathrm{i})(k + \mathrm{i})^{1/2}}{2k^2},$$

and hence write the solution u as a Fourier inversion, being careful to specify the inversion contour.

1. (a) [3 marks] Locally near A,

$$w = -\frac{Q}{2\pi}\log z + c = -\frac{Q}{2\pi}\log r - \frac{\mathrm{i}Q\theta}{2\pi} + c, \quad \boxed{1}$$

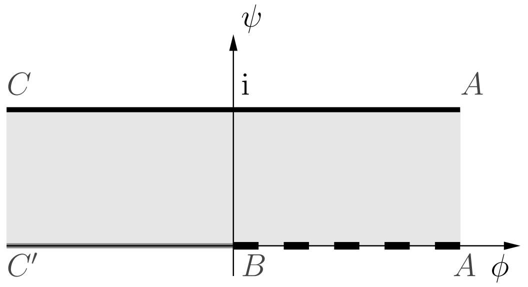
where $z = r\mathrm{e}^{\mathrm{i}\theta}$. Since $\psi = 0$ on AB, where $\theta = \pi/2$, $\mathrm{Im}(c) = Q/4$. Then, on AC, $\psi = \mathrm{Im}(c) = Q/4$. $\boxed{1}$ The flux at infinity is

$$-\frac{Q}{4} = u_\infty h \qquad \Rightarrow \qquad u_\infty = -\frac{Q}{4h}.$$

Now scale x and y with h, u and v with $|u_\infty|$, and ϕ and ψ with $Q/4$. $\boxed{1}$

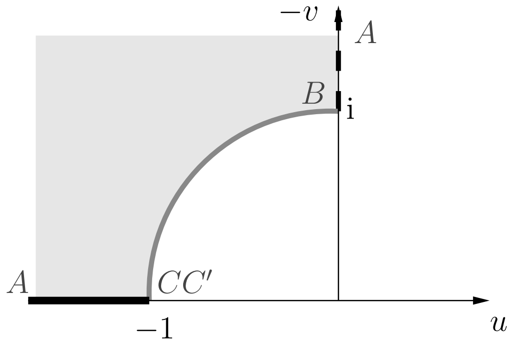
☐ **Bookwork/similar to examples**

(b) [6 marks] Since $\psi = 0$ on ABC' and at infinity the velocity is $(-1, 0) = (\psi_y, -\psi_x)$ with fluid height 1 the stream function takes the value 1 on AC (or from part (a)) $\boxed{1}$. Thus the potential plane is the strip $0 < \psi < 1$, $-\infty < \phi < \infty$, with $\phi(B) = 0$ $\boxed{1}$.



$\boxed{1}$

The velocity at C and C' is $(-1, 0)$. Since $|w'|$ is constant on BC' we must have $|w'| = 1$ on BC', and therefore the velocity at B is $(0, -1)$. $\boxed{1}$ The speed at A tends to infinity $\boxed{1}$. On AC we know $v = 0$. On AB we know $u = 0$.



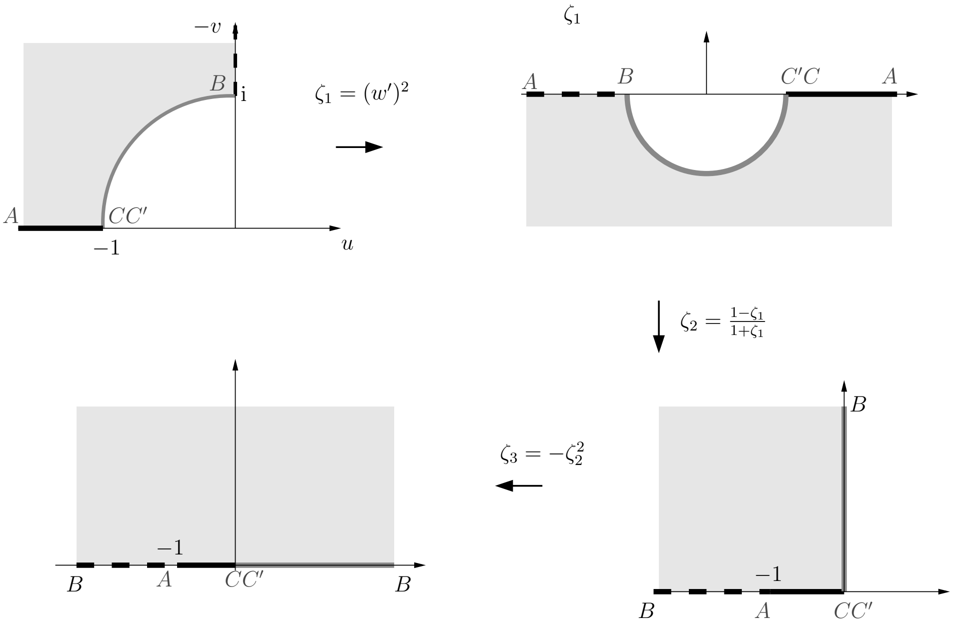
$\boxed{1}$

☐ **New example, but familiar method. New idea having a sink in the flow. Will take some thought to identify the correct region, with the velocity at A tending to infinity.**

(c) [6 marks] For the hodograph plane, first get rid of the corner at infinity by squaring: $\zeta_1 = (w')^2$. This gets us the lower half-plane minus a semi-circle $\boxed{1}$. Then use the mobius map

$$\zeta_2 = \frac{1 - \zeta_1}{\zeta_1 + 1}$$

which sends 1 to 0, -1 to ∞, and ∞ to -1 $\boxed{1}$. Thus this maps the semicircle to the second quadrant $\boxed{1}$. Now square and minus to get the upper half plane $\boxed{1}$.



Thus

$$\zeta_3 = -\left(\frac{1-(w')^2}{1+(w')^2}\right)^2.$$

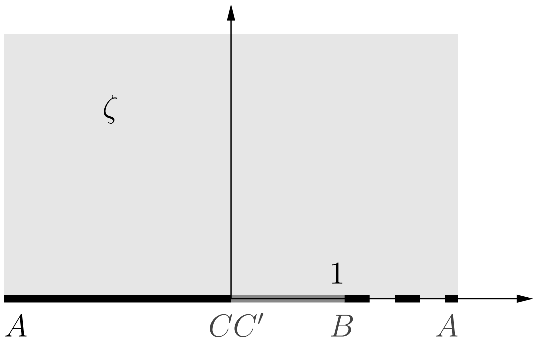
$\zeta_A = -1, \quad \zeta_B = \infty, \;, \quad \zeta_C = 0, \quad \boxed{2}.$

☐ **New example, but some of the maps have been seen before.**

(d) [5 marks] To map the strip in the potential plane to the upper-half plane, we use

$$\zeta = \mathrm{e}^{\pi w}. \quad \boxed{1}$$

This maps B to 1, CC' to 0 and A to ∞. $\boxed{1}$



In the hodograph ζ_3-plane the points are in the wrong places. We need to send 0 to 0, ∞ to 1 and -1 to infinity $\boxed{1}$. Thus we need to set

$$\begin{aligned}\zeta &= \frac{\zeta_3}{\zeta_3+1} = -\left(\frac{1-(w')^2}{1+(w')^2}\right)^2 \Bigg/ \left(1-\left(\frac{1-(w')^2}{1+(w')^2}\right)^2\right) \\ &= -\frac{(1-(w')^2)^2}{(1+(w')^2)^2-(1-(w')^2)^2} = -\frac{(1-(w')^2)^2}{4(w')^2}. \quad \boxed{2}\end{aligned}$$

Thus w satisfies the differential equation

$$\mathrm{e}^{\pi w} = -\frac{(1-(w')^2)^2}{4(w')^2}.$$

☐ **New example. Need to be careful to make sure all the points are in the right place.**

(e) [5 marks] Setting $w' = \mathrm{e}^{-\mathrm{i}\theta}$ gives

$$\mathrm{e}^{\pi w} = -\frac{(1-\mathrm{e}^{-2\mathrm{i}\theta})^2}{4\mathrm{e}^{-2\mathrm{i}\theta}} = -\frac{(\mathrm{e}^{\mathrm{i}\theta}-\mathrm{e}^{-\mathrm{i}\theta})^2}{4} = \sin^2\theta. \quad \boxed{1}$$

Differentiating with respect to θ,

$$2\sin\theta\cos\theta = \pi\mathrm{e}^{\pi w}w'\frac{\mathrm{d}z}{\mathrm{d}\theta} = \pi\sin^2\theta\,\mathrm{e}^{-\mathrm{i}\theta}\frac{\mathrm{d}z}{\mathrm{d}\theta}.$$

Thus

$$\frac{\mathrm{d}z}{\mathrm{d}\theta} = \frac{2\mathrm{e}^{\mathrm{i}\theta}\cos\theta}{\pi\sin\theta}. \quad \boxed{1}$$

Separating into real and imaginary parts

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{2\cos^2\theta}{\pi\sin\theta}, \qquad \frac{\mathrm{d}y}{\mathrm{d}\theta} = \frac{2\cos\theta}{\pi}.$$

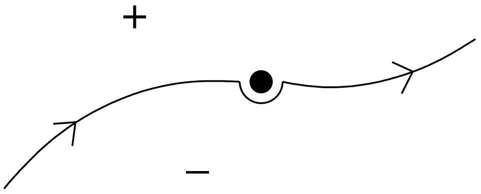
Thus

$$y = 1 + \frac{2\sin\theta}{\pi}, \quad \boxed{1}$$

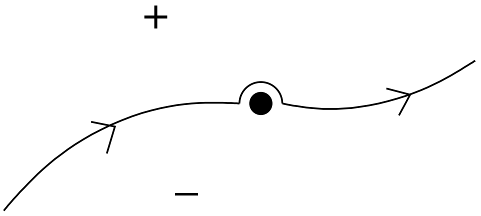
where we have used $y \to 1$ as $\theta \to \pi$ to fix the constant.
At B, $\theta \to 3\pi/2$, $y \to 1 - 2/\pi$. $\boxed{1}$
For a point source the picture would be the same. Simply set $\mathbf{u} \to -\mathbf{u}$. $\boxed{1}$

☐ **New example. Will require some thought**

2. (a) [5 marks] Label the left-hand side of Γ as "+" and the right-hand side as "-" $\boxed{1}$. As $z \to t \in \Gamma$ from the plus side indent the contour with a small semi-circle around t as shown in (a) $\boxed{1}$.



(a)



(b)

Then

$$w_+(t) = \lim_{\epsilon\to 0}\frac{1}{2\pi\mathrm{i}}\left(\int_{\text{one end}}^{t-\epsilon}\frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - t} + \int_{\text{semicircle}|\zeta-t|=\epsilon}\frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - t} + \int_{\text{other end}}^{t-\epsilon}\frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - t}\right).$$

As $\epsilon \to 0$ the semicircle gives a contribution $\frac{1}{2\pi\mathrm{i}} \times \pi\mathrm{i}f(t)$ $\boxed{1}$. Thus

$$w_+(t) = \frac{f(t)}{2} + \frac{1}{2\pi\mathrm{i}}\fint_\Gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - t},$$

where

$$\fint = \lim_{\epsilon\to 0}\left(\int^{t-\epsilon} + \int_{t+\epsilon}\right). \quad \boxed{1}$$

As $z \to t \in \Gamma$ from the minus side we need to indent the contour on the other side with a small semi-circle around t as shown in (b). The semicircle now gives a contribution $-\frac{1}{2\pi\mathrm{i}} \times \pi \mathrm{i} f(t)$ as $\epsilon \to 0$, $\boxed{1}$ so that

$$w_-(t) = -\frac{f(t)}{2} + \frac{1}{2\pi\mathrm{i}} \fint_\Gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - t}.$$

□ **Bookwork**

(b) [5 marks] Let $\tilde{w}(z)$ be holomorphic on $\mathbb{C}\backslash[0,1]$ and such that $\tilde{w}_+(x) = -\tilde{w}_-(x) \neq 0$. Let $W = w/\tilde{w}$. Then

$$W_+(x) - W_-(x) = \frac{w_+(x)}{\tilde{w}_+(x)} - \frac{w_-(x)}{\tilde{w}_-(x)} = \frac{w_+(x) + w_-(x)}{\tilde{w}_+(x)} = \frac{2\mathrm{i}g'(x)}{\tilde{w}_+(x)}.$$

If we write

$$W(z) = \frac{1}{2\pi\mathrm{i}} \int_0^1 \frac{F(\xi)}{\xi - z}\,\mathrm{d}\xi,$$

then, by Plemelj,

$$W_+(x) - W_-(x) = F(x) = \frac{2\mathrm{i}g'(x)}{\tilde{w}_+(x)}.$$

Thus the general solution is

$$W(z) = \frac{1}{\pi} \int_0^1 \frac{g'(\xi)}{\tilde{w}_+(\xi)(\xi - z)}\,\mathrm{d}\xi + H(z),$$

where H is holomorphic on $\mathbb{C}\backslash\{0,1\}$. Thus

$$w(z) = \frac{\tilde{w}(z)}{\pi} \int_0^1 \frac{g'(\xi)}{\tilde{w}_+(\xi)(\xi - z)}\,\mathrm{d}\xi + \tilde{w}(z)H(z),$$

as required.

$\boxed{5}$ **Bookwork**

(c) [5 marks] We choose

$$\tilde{w}(z) = \left(\frac{1-z}{z}\right)^{1/2} \quad \boxed{1}$$

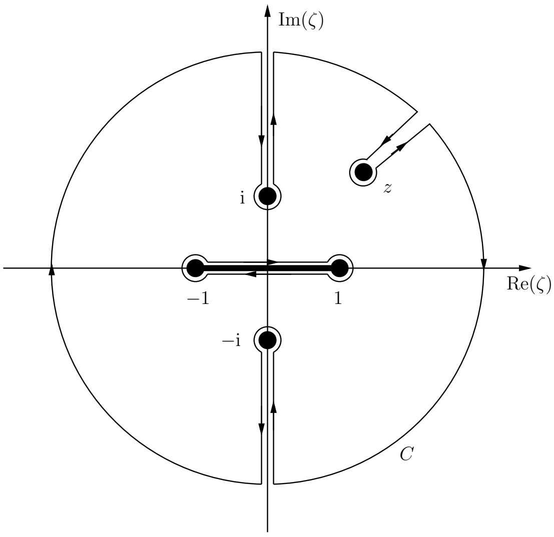
where we choose the branch cut to lie along $[0,1]$ with w real and positive on top of the cut. This means, for example, $0 < \arg(z) < 2\pi$, $-\pi < \arg(1-z) < \pi$. $\boxed{1}$ Then

$$w(z) = \left(\frac{1-z}{z}\right)^{1/2} \frac{1}{\pi} \int_0^1 \frac{\xi^{1/2} g'(\xi)}{(1-\xi)^{1/2}(\xi - z)}\,\mathrm{d}\xi + \left(\frac{1-z}{z}\right)^{1/2} H(z).$$

Since $w = O(z^{-1/2})$ as $z \to 0$ and $w = O(1)$ as $z \to 1$, H can have only removeable singularities there, and is therefore entire. $\boxed{1}$ Since $w = O(1/z)$ as $z \to \infty$, $H = O(1/z)$ as $z \to \infty$. $\boxed{1}$ Hence by Liouville $H \equiv 0$. $\boxed{1}$

□ **Similar to worked examples/bookwork**

(d) [5 marks] With $g(x) = \log(1+x^2)$, $g'(x) = 2x/(1+x^2)$, and



$$
\begin{aligned}
w(z) &= \left(\frac{1-z}{z}\right)^{1/2} \frac{2}{\pi}\int_0^1 \frac{\xi^{3/2}}{(1+\xi^2)(1-\xi)^{1/2}(\xi-z)}\,\mathrm{d}\xi \\
&= \left(\frac{1-z}{z}\right)^{1/2} \frac{1}{\pi}\int_C \frac{\zeta^{3/2}}{(1+\zeta^2)(1-\zeta)^{1/2}(\zeta-z)}\,\mathrm{d}\zeta \\
&= \left(\frac{1-z}{z}\right)^{1/2} 2\mathrm{i}\left(\frac{\mathrm{i}2^{-1/4}\mathrm{e}^{3\mathrm{i}\pi/8}}{2\mathrm{i}(\mathrm{i}-z)} + \frac{-\mathrm{i}2^{-1/4}\mathrm{e}^{5\mathrm{i}\pi/8}}{-2\mathrm{i}(-\mathrm{i}-z)} + \frac{z^{3/2}}{(1+z^2)(1-z)^{1/2}}\right) \\
&= \left(\frac{1-z}{z}\right)^{1/2} 2\mathrm{i}\left(\frac{-(z+\mathrm{i})\mathrm{e}^{\mathrm{i}3\pi/8}}{2^{5/4}(z^2+1)} + \frac{(z-\mathrm{i})\mathrm{e}^{-3\mathrm{i}\pi/8}}{2^{5/4}(z^2+1)} + \frac{z^{3/2}}{(1+z^2)(1-z)^{1/2}}\right) \\
&= \left(\frac{1-z}{z}\right)^{1/2} 2\mathrm{i}\left(\frac{-z(\mathrm{e}^{\mathrm{i}3\pi/8}-\mathrm{e}^{-3\mathrm{i}\pi/8})+\mathrm{e}^{-\mathrm{i}\pi/8}-\mathrm{e}^{\mathrm{i}\pi/8}}{2^{5/4}(z^2+1)} + \frac{z^{3/2}}{(1+z^2)(1-z)^{1/2}}\right) \\
&= \left(\frac{1-z}{z}\right)^{1/2} 2\mathrm{i}\left(\frac{-\mathrm{i}\sin(3\pi/8)z-\mathrm{i}\sin(\pi/8)}{2^{1/4}(z^2+1)} + \frac{z^{3/2}}{(1+z^2)(1-z)^{1/2}}\right) \\
&= \left(\frac{1-z}{z}\right)^{1/2} \frac{2^{3/4}(\sin(\pi/8)+\sin(3\pi/8)z)}{(z^2+1)} + 2\mathrm{i}\frac{z}{(1+z^2)}
\end{aligned}
$$

$\boxed{5}$ **New example. The extra poles in the integral have not been seen before. Getting the right branch of the square root at ±i is crucial and will require some thought**

(e) [5 marks] With

$$w(z) = \frac{1}{2\pi\mathrm{i}}\int_0^1 \frac{f(\xi)}{\xi-z}\,\mathrm{d}\xi$$

by Plemelj

$$w_+(x)+w_-(x) = \frac{1}{\pi\mathrm{i}}\int_0^1 \frac{f(\xi)}{\xi-x}\,\mathrm{d}\xi = 2\mathrm{i}g'(x) = \frac{4\mathrm{i}x}{1+x^2},$$

so that f satisfies the given singular integral equation. A solution is therefore given by

$$f(x) = w_+(x)-w_-(x) = \frac{2^{7/4}(1-x)^{1/2}(\sin(\pi/8)+\sin(3\pi/8)x)}{x^{1/2}(x^2+1)}.$$

$\boxed{5}$ **New example. Will require some thought**

3. (a) [3 marks] $\bar{f}_+(k)$ is defined for $\mathrm{Im}(k) > 0$. $\boxed{1}$

$$\bar{f}_+(k) = \int_0^\infty x\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x = \left[\frac{x\mathrm{e}^{\mathrm{i}kx}}{\mathrm{i}k}\right]_0^\infty - \int_0^\infty \frac{\mathrm{e}^{\mathrm{i}kx}}{\mathrm{i}k}\,\mathrm{d}k = -\left[\frac{\mathrm{e}^{\mathrm{i}kx}}{(\mathrm{i}k)^2}\right]_0^\infty = -\frac{1}{k^2}. \quad \boxed{1}$$

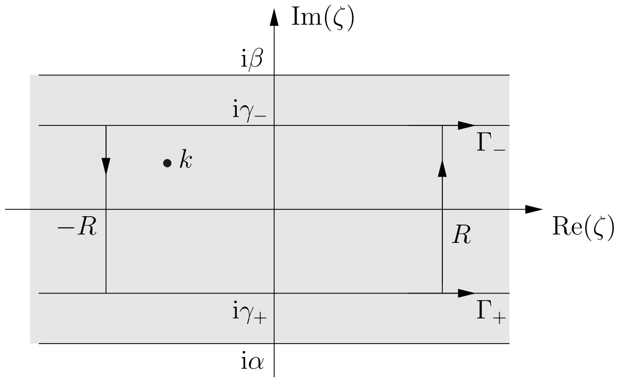
It may be analytically continued to $\mathbb{C}\backslash\{0\}$. $\boxed{1}$

☐ **Bookwork/Straightforward calculation**

(b) [5 marks] By Cauchy

$$G(k) = \frac{1}{2\pi\mathrm{i}}\int_\Gamma \frac{G(\zeta)}{\zeta - k}\,\mathrm{d}\zeta, \quad \boxed{1}$$

where $\Gamma \subset \Omega$ is the rectangular contour with sides $\mathrm{Re}(\zeta) = \pm R$, $\mathrm{Im}(\zeta) = \gamma_\pm$ where $-R < \mathrm{Re}(k) < R$, $\alpha < \gamma_+ < \mathrm{Im}(k) < \gamma_- < \beta$. $\boxed{1}$



Since $G(\zeta) \to 0$ as $|\zeta| \to \infty$ in Ω the contribution from the vertical sides $\to 0$ as $R \to \infty$, $\boxed{1}$ giving $G(k) = G_+(k) - G_-(k)$ for $\gamma_+ < \mathrm{Im}(k) < \gamma_-$ where

$$G_\pm(k) = \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_\pm} \frac{G(\zeta)}{\zeta - k}\,\mathrm{d}\zeta,$$

and $\Gamma_\pm = \{\xi + \mathrm{i}\gamma_\pm : -\infty < \xi < \infty\}$. $\boxed{1}$ Note that Γ_+ passes underneath k, while Γ_- passes above k. Since $G_\pm(k)$ is holomorphic everywhere except on $\Gamma_\pm$, we deduce that $G_+(k)$ is holomorphic in $\mathrm{Im}(k) > \gamma_+$ and $G_-(k)$ is holomorphic in $\mathrm{Im}(k) < \gamma_-$. $\boxed{1}$

☐ **Bookwork**

(c) [7 marks] Put the branch cuts up the the imaginary axis from i and down the imaginary axis from −i, so that $-3\pi/2 < \arg(k - \mathrm{i}) < \pi/2$ and $-\pi/2 < \arg(k + \mathrm{i}) < 3\pi/2$. Then $-\pi/2 < \arg(k - \mathrm{i})^{1/2} < 0$ and $0 < \arg(k + \mathrm{i})^{1/2} < \pi/2$ for $-1 < \mathrm{Im}(k) < 1$. $\boxed{2}$

$$G_+(k) = \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_+} \frac{(\zeta - \mathrm{i})^{1/2}}{\zeta^2(\zeta - k)}\,\mathrm{d}\zeta, \qquad 0 < \gamma_+ < \mathrm{Im}(k)$$

Since there is a branch point at $\zeta = \mathrm{i}$ close at $-\mathrm{i}\infty$ to give $-2\pi\mathrm{i}\times$ residue at $\zeta = 0$. $\boxed{1}$ For this we need the Taylor series

$$\frac{(\zeta - \mathrm{i})^{1/2}}{\zeta - k} = \frac{(-\mathrm{i})^{1/2}}{-k}\frac{(1 + \mathrm{i}\zeta)^{1/2}}{(1 - \zeta/k)} = \frac{(-\mathrm{i})^{1/2}}{-k}\left(1 + \left(\frac{\mathrm{i}}{2} + \frac{1}{k}\right)\zeta + \cdots\right) + \cdots \quad \boxed{1}$$

Thus

$$G_+(k) = \frac{(-\mathrm{i})^{1/2}}{k}\left(\frac{\mathrm{i}}{2} + \frac{1}{k}\right) = \frac{(-\mathrm{i})^{1/2}(\mathrm{i}k + 2)}{2k^2} = \frac{\mathrm{e}^{-\mathrm{i}\pi/4}\mathrm{i}(k - 2\mathrm{i})}{2k^2} = \frac{\mathrm{e}^{\mathrm{i}\pi/4}(k - 2\mathrm{i})}{2k^2}. \quad \boxed{2}$$

$$G_-(k) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_-} \frac{(\zeta - \mathrm{i})^{1/2}}{\zeta^2 (\zeta - k)} \, \mathrm{d}\zeta, \qquad \mathrm{Im}(k) < \gamma_- < 1$$

Again close at $-\mathrm{i}\infty$ to give the same contribution from $\zeta = 0$ plus the residue at $\zeta = k$:

$$G_-(k) = \frac{\mathrm{e}^{\mathrm{i}\pi/4}(k - 2\mathrm{i})}{2k^2} - \frac{(k-\mathrm{i})^{1/2}}{k^2}. \quad \boxed{1}$$

☐ **Will require some thought because of the double pole.**

(d) [5 marks] Taking the Fourier transform of the equation gives

$$-k^2 \bar{u} + \bar{u}_{yy} - \bar{u} = 0,$$

so that

$$\bar{u} = A(k) \mathrm{e}^{-(k^2+1)^{1/2} y} + B(k) \mathrm{e}^{(k^2+1)^{1/2} y}, \quad \boxed{1}$$

where the branch cuts of $(k^2+1)^{1/2}$ are up the imaginary axis from i and down the imaginary axis from $-\mathrm{i}$, with $(k^2+1)^{1/2}$ real and positive for real k. $\boxed{1}$ Since $u \to 0$ as $y \to \infty$ we must have $B = 0$. $\boxed{1}$ The boundary conditions give

$$\begin{aligned} A(k) &= \bar{f}_- + \bar{f}_+, \\ -(k^2+1)^{1/2} A(k) &= \bar{g}_+, \quad \boxed{1} \end{aligned}$$

so that

$$\bar{g}_+ = -(k^2+1)^{1/2} \left(\bar{f}_- + \bar{f}_+ \right) = -(k^2+1)^{1/2} \left(\bar{f}_- - \frac{1}{k^2} \right),$$

i.e.

$$\bar{g}_+ + (k^2+1)^{1/2} \bar{f}_- = \frac{(k^2+1)^{1/2}}{k^2}. \quad \boxed{1}$$

☐ **New example, but standard technique**

(e) [5 marks] Factoring $(k^2+1)^{1/2} = (k+\mathrm{i})^{1/2}(k-\mathrm{i})^{1/2}$ gives

$$\frac{\bar{g}_+}{(k+\mathrm{i})^{1/2}} = -(k-\mathrm{i})^{1/2} \bar{f}_- + \frac{(k-\mathrm{i})^{1/2}}{k^2}. \quad \boxed{1}$$

We need an additive decomposition of the last term, which is given by part (c). Then

$$\frac{\bar{g}_+}{(k+\mathrm{i})^{1/2}} - \frac{\mathrm{e}^{\mathrm{i}\pi/4}(k-2\mathrm{i})}{2k^2} = -(k-\mathrm{i})^{1/2} \bar{f}_- - \frac{\mathrm{e}^{\mathrm{i}\pi/4}(k-2\mathrm{i})}{2k^2} + \frac{(k-\mathrm{i})^{1/2}}{k^2}. \quad \boxed{1}$$

The LHS is holomorphic in $\mathrm{Im}(k) > 0$, while the RHS is holomorphic in $\mathrm{Im}(k) < 1$. Since they agree on the strip $0 < \mathrm{Im}(k) < 1$ both can be extended to an entire function. $\boxed{1}$ Since $k^{-1/2} \bar{g}_+ \to 0$ as $k \to \infty$ this entire function must be zero. Thus

$$\bar{g}_+ = \frac{\mathrm{e}^{\mathrm{i}\pi/4}(k-2\mathrm{i})(k+\mathrm{i})^{1/2}}{2k^2}.$$

Then

$$A(k) = -\frac{\mathrm{e}^{\mathrm{i}\pi/4}(k-2\mathrm{i})}{2k^2(k-\mathrm{i})^{1/2}}.$$

Thus

$$u = -\frac{\mathrm{e}^{\mathrm{i}\pi/4}}{4\pi} \int_\Gamma \frac{(k-2\mathrm{i})}{k^2(k-\mathrm{i})^{1/2}} \mathrm{e}^{-\mathrm{i}kx - (k^2+1)^{1/2} y} \, \mathrm{d}k, \quad \boxed{1}$$

where the contour must pass above $k = 0$ but below $k = \mathrm{i}$. $\boxed{1}$

☐ **New example. Will require some thought**