A12204W1

SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part C: Paper C5.2

ELASTICITY AND PLASTICITY

TRINITY TERM 2017

THURSDAY, 1 JUNE 2017, 9.30am to 11.15am

You may submit answers to as many questions as you wish but only the best two will count for the total mark. All questions are worth 25 marks.

You should ensure that you:

- *start a new answer booklet for each question which you attempt.*
- *indicate on the front page of the answer booklet which question you have attempted in that booklet.*
- *cross out all rough working and any working you do not want to be marked. If you have used separate answer booklets for rough work please cross through the front of each such booklet and attach these answer booklets at the back of your work.*
- *hand in your answers in numerical order.*

If you do not attempt any questions, you should still hand in an answer booklet with the front sheet completed.

Do not turn this page until you are told that you may do so

1. You are given Cauchy's momentum equation in component form

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \tau_{ij}}{\partial x_j}$$

where $\mathcal{T} = (\tau_{ij})$ is the stress tensor, $\mathbf{u}$ is the displacement field and ρ is the (constant) density of the solid. (Throughout this question you should neglect the role of body forces such as gravity.)

(a) [6 marks]

(i) Use Cauchy's equation to show that for any fixed volume V,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V \left(\tfrac{1}{2}\rho \left| \frac{\partial \mathbf{u}}{\partial t} \right|^2 + \mathcal{W} \right) \mathrm{d}V = -\int_{\partial V} \mathcal{F} \cdot \mathbf{n} \; \mathrm{d}S \qquad (1)$$

where $\mathcal{F}_j = -\tau_{ij} \partial u_i / \partial t$ is the energy flux, $\mathcal{W}(e_{ij})$ is the strain energy density (a function of the strain tensor $\mathcal{E} = (e_{ij})$, such that $\tau_{ij} = \partial \mathcal{W} / \partial e_{ij}$).

(ii) Interpret the individual terms in (1), and the equation as a whole.

(b) [4 marks] The remainder of this question concerns wave propagation between two semi-infinite elastic materials (welded together at $y = 0$) in anti-plane strain. For $y > 0$ the shear modulus is μ_+ while for $y < 0$ the shear modulus is μ_-. All other properties of the materials are identical.

(i) In anti-plane strain, the displacement field takes the form $\mathbf{u}(x, y, z, t) = w(x, y, t)\mathbf{e}_z$. Show that for elastic waves in anti-plane strain, $w(x, y, t)$ satisfies a two-dimensional wave equation with a wave speed that you should give in terms of the Lamé parameters, λ and μ, as well as the density ρ.

(ii) Write down the appropriate boundary conditions on $w(x, y, t)$ at $y = 0$.

(c) [9 marks] A wave with wavenumber k_- and angular frequency ω is incident on the boundary $y = 0$ from $y = -\infty$ at an angle α to the y-axis. The incident wave thus corresponds to a displacement $\mathbf{u}_{\text{inc}} = w_{\text{inc}}(x, y, t)\mathbf{e}_z$ with

$$w_{\text{inc}}(x, y, t) = \mathrm{Re}\left\{ \exp\left[\mathrm{i}k_-(x \sin\alpha + y \cos\alpha) - \mathrm{i}\omega t \right] \right\}.$$

(i) Write down expressions for the reflected wave (which propagates in $y < 0$ at an angle β to the y-axis) and the transmitted wave (which propagates in $y > 0$ at an angle γ to the y-axis). Hence derive expressions for the wavenumber k_+ of the transmitted wave and the angle γ.

(ii) Show that the amplitudes of the reflected and transmitted waves, R and T respectively, may be written

$$R = \frac{\mu_- \cot\alpha - \mu_+ \cot\gamma}{\mu_- \cot\alpha + \mu_+ \cot\gamma}, \quad T = \frac{2\mu_- \cot\alpha}{\mu_- \cot\alpha + \mu_+ \cot\gamma}.$$

(d) [6 marks] Now consider specifically the case $\mu_+ > \mu_-$.

(i) What happens if $\alpha > \alpha_c = \sin^{-1}(\sqrt{\mu_-/\mu_+})$?

(ii) By letting $\gamma = \pi/2 - \mathrm{i}\theta$ and the amplitude $T = |T|e^{\mathrm{i}\phi}$, or otherwise, calculate $\langle \mathcal{F} \rangle$ for $\alpha > \alpha_c$.

[Here $\langle f \rangle = T^{-1} \int_t^{t+T} f(s) \; \mathrm{d}s$ *denotes the time average of a periodic function* $f(t)$ *with period* T*.* $\mathcal{F}$ *is as defined in part (a).]*

What is the (time-averaged) flux of energy to $y = +\infty$?

(iii) Show that when $\alpha > \alpha_c$ displacements within $y > 0$ decay over a typical vertical distance

$$k_-^{-1} \left[\sin^2\alpha - \mu_-/\mu_+ \right]^{-1/2}.$$

2. This question concerns the contact between a light elastic membrane and a massive rigid object. Throughout the question, you may neglect the mass of the membrane, even when you account for the mass of the rigid object.

(a) [11 marks] A semi-infinite 2-D strip membrane is clamped at $x = \pm L$ and is subject to a (y-independent) loading, $p(x)$, in the negative z-direction. The membrane is stretched by a constant tension T in the x-direction. The shape of the membrane may be described by $z = w(x)$ for $-L \leqslant x \leqslant L$ with $w(\pm L) = 0$.

(i) Show that small transverse displacements, $w(x)$, satisfy

$$T\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = p. \tag{2}$$

If the membrane is brought into contact with a rigid obstacle $z = f(x)$, show that T and $\mathrm{d}w/\,\mathrm{d}x$ are continuous at the points where the membrane makes contact with the obstacle.

(ii) A cylindrical object of mass m per unit length, and radius R is laid to rest on the stretched membrane. Its weight per unit length, mg, deforms the membrane, such that the lowest point of the membrane lies a vertical distance δ below the edges of the membrane (at $x = \pm L$). You should assume that the shape of the cylinder is then approximated by $z = f(x) = -\delta + x^2/(2R)$.
Determine the contact set, $[-s, s]$, as δ varies, but subject to $\delta R/L^2 \ll 1$.
Show that the cylinder rests in equilibrium with

$$\delta \approx \frac{mgL}{2T}. \tag{3}$$

(b) [8 marks] Consider now a membrane that is clamped at a circular boundary, $r = L$, with r the usual polar coordinate. (The shape of the membrane may then be written $z = w(r)$ with the clamping condition written $w(L) = 0$.)

(i) Assuming that the tension T within the membrane remains constant, determine the generalization of (2) to the axisymmetric problem.
What are the appropriate conditions on $w(r)$ at the edge of a contact region?

(ii) A sphere of mass m and radius R is laid to rest on the stretched membrane. Its weight, mg, deforms the membrane, such that the lowest point of the membrane lies a vertical distance δ below the edges of the membrane (at $r = L$). You should assume that the shape of the sphere may be approximated by $z = f(r) = -\delta + r^2/(2R)$.
Determine the contact set, $[0, s]$, as δ varies, and find an expression for δ as a function of the weight of the sphere.

(c) [6 marks] A simple model for a trampoline bounce makes use of the results of part (a), to determine how the lowest position of the bouncer, $\delta(t)$, evolves. Assume that the shape of the trampoline is determined instantaneously for a given $\delta(t)$; in particular, the restoring force from the trampoline on the bouncer is assumed to be a function of $\delta(t)$ only.

(i) Using results from part (a) as appropriate, write down, and solve, Newton's second law for the evolution of $\delta(t)$ from the moment of first contact (at $t = 0$). (Denote the initially downward speed of the bouncer by V.)

(ii) Show that the maximum vertical stretching of the trampoline is attained at time

$$t_{\max} = \frac{\pi}{\omega} - \frac{1}{\omega}\tan^{-1}\frac{V\omega}{g} \tag{4}$$

where $\omega = (2T/mL)^{1/2}$.

(iii) Determine the duration of the contact between the bouncer and the trampoline, and compare this time to the natural period of the motion, $2\pi/\omega$, in the limits $V\omega/g \ll 1$ and $V\omega/g \gg 1$.

3. A thin elliptical Mode III crack, whose boundary $\partial\Omega$ is given by

$$\frac{x^2}{c^2\cosh^2\epsilon} + \frac{y^2}{c^2\sinh^2\epsilon} = 1,$$

is subject to an antiplane strain displacement field, $\mathbf{u} = w(x,y)\mathbf{e}_z$.

(a) [12 marks] A shear stress $\tau_{yz} = \sigma$ is applied in the far field.

(i) Justify the conditions

$$\frac{\partial w}{\partial n} = 0 \quad \text{on} \quad \partial\Omega$$

and $w \sim \sigma y/\mu$ as $x^2 + y^2 \to \infty$.

(ii) Show that the Joukowsky transformation, $x + \mathrm{i}y = z = \frac{1}{2}c(\zeta + \zeta^{-1})$ conformally maps the region $|\zeta| > e^{\epsilon}$ ($\epsilon > 0$) to the outside of the crack.

(iii) What is the inverse map from z to ζ?

(iv) Introducing polar coordinates (r,θ) such that $\zeta = re^{\mathrm{i}\theta}$, show that

$$w = \frac{c\sigma}{2\mu}\mathrm{Im}\left\{\zeta - \frac{e^{2\epsilon}}{\zeta}\right\}.$$

[*You may use a heuristic justification of the appropriate boundary condition at* $r = e^{\epsilon}$, *based on your answer to part (i).*]

(b) [9 marks] With the crack aligned as before, the boundary tension is now applied at an angle α to the horizontal, so that $(\tau_{xz}, \tau_{yz}) \sim \sigma(\cos\alpha, \sin\alpha)$ as $x^2 + y^2 \to \infty$.

(i) Repeat the analysis of part (a) to find the displacement in this case.

(ii) Give an expression for the displacement $w(z)$ in the limit $\epsilon \to 0$.

(iii) Does the rotation of the applied load increase or decrease the intensity of the singularity that is observed at the crack tips?
[*You may find it helpful to note that* $\zeta^{-1} = 2z/c - \zeta$.]

(c) [4 marks] Return to the case $\alpha = \pi/2$, and consider $0 < \epsilon \ll 1$.

(i) Show that the radius of curvature of the crack tip, $r_0 \sim \epsilon^2 c$ for $\epsilon \ll 1$.

(ii) Show that as the crack tip is approached from within the material, e.g. as $z \searrow c$, the stress $\tau_{yz} \sim \sigma(c/r_0)^{1/2}$.
[*You may find it useful to note that if a displacement field can be written as* $w = Im\{f(z)\}$, *then* $\tau_{yz} = \mu Re\{f'(z)\}$.]

Draft Marks

Q1

C5.2 Model Solutions

Comm

B

a)

i) We are given the equation of motion:

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \tau_{ij}}{\partial x_j}$$

so that:

$$\rho \frac{\partial u_i}{\partial t}\frac{\partial^2 u_i}{\partial t^2} = \frac{\partial u_i}{\partial t}\frac{\partial \tau_{ij}}{\partial x_j}$$

and so:

$$\frac{d}{dt}\int_V \frac{1}{2}\rho\left|\frac{\partial u}{\partial t}\right|^2 dV = \int_V \frac{\partial}{\partial x_j}\left(\tau_{ij}\frac{\partial u_i}{\partial t}\right) dV - \int_V \tau_{ij}\frac{\partial}{\partial t}\left(\frac{\partial u_i}{\partial x_j}\right) dV$$

Using symmetry of τ_{ij}, we have:

$$\tau_{ij}\frac{\partial}{\partial t}\left(\frac{\partial u_i}{\partial x_j}\right) = \tau_{ij}\frac{\partial}{\partial t}e_{ij} = \frac{\partial W}{\partial t}$$

where W is such that: $\tau_{ij} = \frac{\partial W}{\partial e_{ij}}$.

Hence:

$$(*) \qquad \frac{d}{dt}\int_V \frac{1}{2}\rho\left|\frac{\partial u}{\partial t}\right|^2 + W\, dV = \int_V \frac{\partial}{\partial x_j}\left(\tau_{ij}\frac{\partial u_i}{\partial t}\right) dV$$

$$= \int_{\partial V} \tau_{ij}\frac{\partial u_i}{\partial t} n_j \, dS$$

$$= -\int_{\partial V} \underline{F}\cdot\underline{n}\, dS$$

where $F_j = -\tau_{ij}\frac{\partial u_i}{\partial t}$.

5

ii) $\underline{F}$ is the energy flux, so that $(*)$ represents conservation of energy (rate of change of elastic + kinetic energy = $-$flux through boundary).

6

Standard material covered in lectures. Definition of flux F new here.

1.2

1 Mks | b)

(i) Returning to the eqn of motion, we note that in anti-plane strain, the only non-zero components of the stress tensor are:

Comment: B — Several examples with anti-plane strain

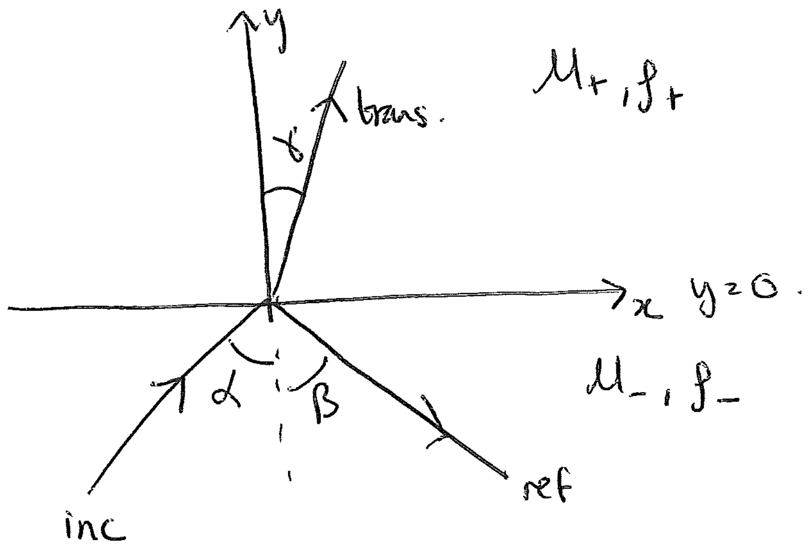
$$\tau_{xz} = \mu \frac{\partial w}{\partial x}$$

$$\text{and } \tau_{yz} = \mu \frac{\partial w}{\partial y}$$

Hence: $$\rho \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left(\mu \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu \frac{\partial w}{\partial y}\right)$$

or $$\frac{\partial^2 w}{\partial t^2} = c^2 \nabla_H^2 w \quad \text{with } \nabla_H^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\text{and } c^2 = \mu/\rho .$$



ii) Boundary conditions at $y=0$ are that

$$[w]_-^+ = 0 \qquad [\text{cty of displacements}]$$

$$\text{and } \left[\mu \frac{\partial w}{\partial y}\right]_-^+ = 0 \qquad [\text{cty of } \tau_{yz}].$$

B — Examples of this BC in lectures

c)

i) $$\underline{u}_{inc} = \underline{e}_z \exp\left[i k_-(x\sin\alpha + y\cos\alpha) - i\omega t\right]$$ (Re part understood)

$$\underline{u}_{ref} = \underline{e}_z R \exp\left[i k_-(x\sin\beta - y\cos\beta) - i\omega t\right]$$

$$\underline{u}_{trans} = \underline{e}_z T \exp\left[i k_+(x\sin\gamma + y\cos\gamma) - i\omega t\right].$$

S

where: $\omega^2 \frac{\mu_\pm}{\rho_\pm} = k_\pm^2$

$\Rightarrow \boxed{k_\pm = \omega/c_\pm}$

mk marks

To have cty of w @ $y=0$, must have same x dependence in each part, ie:

$$k_- \sin\alpha = k_- \sin\beta = k_+ \sin\gamma$$

$$\therefore \quad \beta = \alpha \qquad \text{(reflection is specular)}.$$

and also:

$$\frac{\omega}{c_+}\sin\gamma = \cancel{\omega}\,\frac{\sin\alpha}{c_-}$$

$$\Rightarrow \sin\gamma = \frac{c_+}{c_-}\sin\alpha \qquad \text{(Snell's law)}.$$

3

ii) To determine amplitudes T and R, use cty bcs @ $y=0$:

$$[w]_-^+ = 0 \Rightarrow 1+R = T$$

$$\left[\mu\frac{dw}{dy}\right]_-^+ = 0 \Rightarrow \mu_+ T.\, k_+\cos\gamma = \cancel{\mu_- R\, k_- \cos\beta}\ \mu_-(1-R)k_-\cos\alpha$$

$$\Rightarrow T = \frac{\mu_-}{\mu_+}(1-R)\underbrace{\frac{k_-\cos\alpha}{k_+\cos\gamma}} = 1+R.$$

$$\Rightarrow R = \frac{\mu_-\cot\alpha - \mu_+\cot\gamma}{\mu_-\cot\alpha + \mu_+\cot\gamma}, \quad T = \frac{2\mu_-\cot\alpha}{\mu_+\cot\gamma + \mu_-\cot\alpha}.$$

4

c) i) Snell's law shows that there's a problem when $\sin\gamma > 1$

ie $\sin\alpha > c_-/c_+$, $\alpha > \underbrace{\sin^{-1}(c_-/c_+)}_{=\alpha_c}$

This is total internal reflection.

1

1.4

Comments

N

To make further progress, for $\alpha > \alpha_c$, ~~note that~~ let:

ii)

?

$$\gamma = \pi/2 - i\theta$$

Then: $\cos\gamma = i\sinh\theta$, $\sin\gamma = \cosh\theta$.

and Snell's Law becomes:

$$\cosh\theta = \frac{c_+}{c_-}\sin\alpha = \left(\frac{\mu_+}{\mu_-}\frac{\rho_-}{\rho_+}\right)^{1/2}\sin\alpha .$$

Total internal reflection not covered (beyond vague statement about evanescent waves)

ii) Now, $F_j = -\tau_{ij}\frac{\partial u_i}{\partial t} \Rightarrow \underline{F} = -\mu\frac{\partial w}{\partial y}\frac{\partial w}{\partial t}\underline{e}_y - \mu\frac{\partial w}{\partial x}\frac{\partial w}{\partial t}\underline{e}_x$.

Letting $T = |T|e^{i\varphi}$, we have:

$$\frac{\langle \underline{E}\rangle}{-\mu} = \cancel{\ldots}\,\cancel{(ik_+\cosh\theta)}\Big\langle \mathrm{Re}\Big[ik_+\cosh\theta\,|T|\exp\big[i\varphi + ik_+x\sin\gamma - k_+y\sinh\theta - i\omega t\big]\Big]$$

$$\cdot\,\mathrm{Re}\Big[-i\omega\,|T|\exp[\;]\Big]\Big\rangle\underline{e}_x$$

$$+\Big\langle \mathrm{Re}\Big[-k_+\sinh\theta\,|T|\exp[\;]\Big]\cdot \times\,\mathrm{Re}\Big[-i\omega\,|T|\exp[\;]\Big]\Big\rangle\underline{e}_y$$

these two out of phase $\Rightarrow \langle\,\rangle = 0$

Calculation of flux new

$$\Rightarrow \langle \underline{E}\rangle = \frac{\mu}{2}k_+\omega|T|^2\exp(-2k_+y\sinh\theta)\cosh\theta\,\underline{e}_x .$$

so as $y \to +\infty$, $\langle \underline{E}\rangle \to 0$ (no energy is emitted to $+\infty$).

4

iii) Energy confined within layer of typical thickness:

1

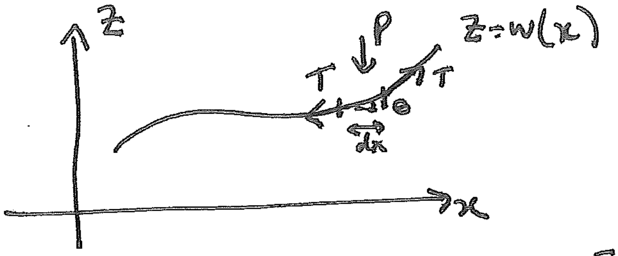
$$\frac{1}{k_+\sinh\theta} = \frac{1}{k_-\frac{c_-}{c_+}\sqrt{\frac{c_+^2}{c_-^2}\sin^2\alpha - 1}} = \frac{1}{k_-\sqrt{\sin^2\alpha - c_-^2/c_+^2}}.$$

Calculation of skin depth is new.

Draft Marks

Q2

9/(i)



Balancing forces on a small segment of length $ds = dx/\cos\theta$.

$$\left[\begin{pmatrix} T\cos\theta \\ T\sin\theta \end{pmatrix}\right]_x^{x+\delta x} + \begin{pmatrix} 0 \\ -p\delta x \end{pmatrix} = \underline{0}$$

For small $|w'|$, $\theta \ll 1 \Rightarrow \cos\theta \simeq 1$

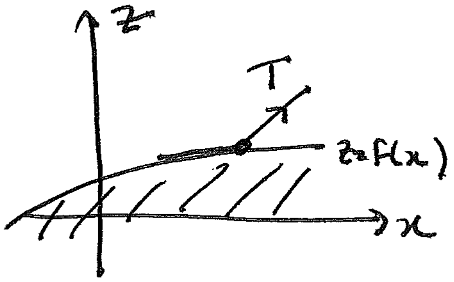
$\sin\theta \simeq dw/dx$

$s \simeq x$

$$\Rightarrow \frac{dT}{dx} = 0 \quad \text{and} \quad \boxed{T\frac{d^2w}{dx^2} = p}$$

3

Similarly, at a point where the string makes smooth contact with the rigid object:



Force balance gives:

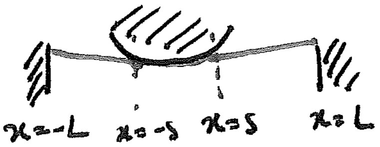
$$\left[\begin{pmatrix} T\cos\theta \\ T\sin\theta \end{pmatrix}\right]_-^+ = 0$$

$$\Rightarrow \boxed{[T]_-^+ = [Tw']_-^+ = 0.}$$

3

(ii)

In contact, have $w = -\delta + x^2/2R$, out of contact we have:



$$T\frac{d^2w}{dx^2} = 0$$

$$\Rightarrow w = Ax + B$$

Consider $x > 0$ (by symmetry) $\Rightarrow w = A(x-L)$. ($\because w(L) = 0$)

Continuity conditions

$[w]_{s^-}^{s^+} = 0 \Rightarrow -\delta + s^2/2R = A(s-L)$ (1)

$[w']_{s^-}^{s^+} = 0 \Rightarrow s/R = A$

Comm

B

B

Comments

Then we have (1) $\Rightarrow \quad -\delta + s^2/2R = \frac{s}{R}(s-L)$

$$\Rightarrow -\delta = \frac{s^2}{2R} - \frac{sL}{R}$$

or: $\quad s^2 - 2sL + 2\delta R = 0.$

Hence: $\quad s = \frac{1}{2}\left[2L \pm \sqrt{4L^2 - 8\delta R}\right]$

choose −ve root to have $s<L$ (or $s \to 0$ as $\delta \to 0$)

For $\delta R/L^2 \ll 1$, $\quad s \simeq \frac{1}{2}\left[\cancel{2L} - 2L\left(\cancel{1} - \frac{2\delta R}{L^2}\cdot\frac{1}{2}\right)\right]$

$$= \delta R/L.$$

To calculate the force, note that the reaction force $N = -T\frac{d^2w}{dx^2}$

Hence $N = -T/R$ and: $\quad F = \int_{-s}^{s} |N|\,dx$

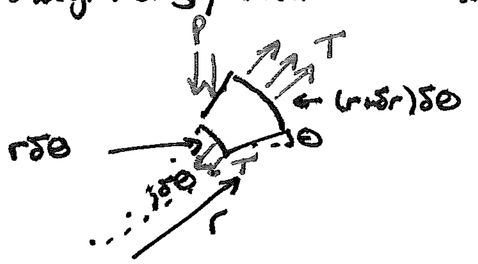
$$= 2sT/R$$

But from the above, $\quad s = \delta R/L$

5 $\Rightarrow \boxed{F = 2T\delta/L}$

or $\boxed{\delta = \frac{F.L}{2T}}$ where $F = mg$ is the weight per unit length of cylinder

b) (i) In axisymmetry, consider a segment $\delta\theta$, between r and $r+\delta r$.



Assuming T constant and considering vertical force balance:

$$T\,(r+\delta r)\delta\theta.\ \sin\theta(r+\delta r) - T.(r\delta\theta)\sin\theta(r) - p.(r\delta\theta).\delta r = 0$$

$\sin\theta \simeq dw/dr$ (small slopes)

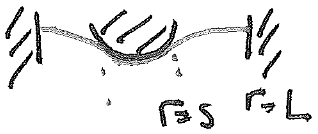
$$\Rightarrow \quad T\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right) = p$$

3 Force balance @ contact line $\Rightarrow \left[\frac{dw}{dr}\right]_-^+ = 0.$

Problem of contact of a rod and 2D membrane done in lectures

(ii) Now, in contact we have $w = -\delta + r^2/2R$

Out of contact, $T \frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right) = 0$



$\Rightarrow w = A \log r/L \qquad (\text{since } w(L) = 0)$

$[w]_{s^-}^{s^+} = 0 \Rightarrow A \log s/L = -\delta + s^2/2R$

$[w']_{s^-}^{s^+} = 0 \Rightarrow s/R = A/s \Rightarrow A = s^2/R$

$$\therefore \quad \delta = \frac{s^2}{R}\left[\frac{1}{2} + \log L/s\right] = \frac{s^2}{R}\log\left(\frac{e^{1/2}L}{s}\right).$$

As before, $N = -T\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)$ with $w = -\delta + r^2/2R$

$= -2T/R$

so then force applied is $F = \int_0^s 2\pi r.|N|\,dr$

$= 2\pi T s^2/R$

$$\Rightarrow s = \left(\frac{FR}{2\pi T}\right)^{1/2}$$

and hence: $\delta = \frac{F\cancel{R}}{2\pi T \cdot \cancel{R}} \log \frac{e^{1/2}L}{(FR/2\pi T)^{1/2}}$

$$\therefore \quad \boxed{\delta = \frac{F}{4\pi T}\log\frac{2\pi e T L^2}{FR}}$$

Then substitute F=mg.

S

c/ Neglecting weight and inertia of string, we have that

$m.\ddot{\delta} = mg - F$

$= mg - 2T\delta/L$

$\Rightarrow \ddot{\delta} = g - \omega^2\delta$ where $\omega = \sqrt{\frac{2T}{mL}}$.

We have initial conditions $\delta(0) = 0$

$\dot{\delta}(0) = V$

N

2.4

So solution is: $\delta = \frac{g}{\omega^2}(1-\cos\omega t) + \frac{V}{\omega}\sin\omega t$

$$\dot{\delta} = \frac{g}{\omega}\sin\omega t + V\cos\omega t$$

δ is maximized when $\dot{\delta}=0$ ie @ $t = t_{max}$:

$$\tan\omega t_{max} = -\frac{V\omega}{g}$$

Hence $t_{max} = \frac{\pi}{\omega} - \frac{1}{\omega}\tan^{-1}\frac{V\omega}{g}$

and then: $\delta_{max} = \delta(t_{max})$

$$= \frac{g}{\omega^2}\left[1 + \frac{1}{\sqrt{1+V^2\omega^2/g^2}} + \frac{\omega V}{g}\cdot\frac{\omega V/g}{(1+V^2\omega^2/g^2)^{1/2}}\right]$$

$$= \frac{g}{\omega^2}\left[1+\sqrt{1+\frac{\omega^2V^2}{g^2}}\right].$$

[Require $\frac{\delta_{max} R}{L^2} \ll 1$, i.e. $\frac{g R m \cancel{L}}{L^{\cancel{2}}\cdot 2T\cancel{g}}\left[1+\sqrt{1+\frac{2TV^2}{mg^2L}}\right] \ll 1$]

Find that time in contact is $t_{contact}$ s.t. $\delta(t_{contact}) = 0$

i.e. $$0 = \frac{g}{\omega^2}\underbrace{(1-\cos\omega t_c)}_{2\sin^2\frac{\omega t_c}{2}} + \frac{V}{\omega}\underbrace{\sin\omega t_c}_{2\sin\frac{\omega t_c}{2}\cos\frac{\omega t_c}{2}}$$

$$\Rightarrow \tan\frac{\omega t_c}{2} = -\frac{\omega V}{g}$$

and hence $t_{contact} = \frac{2\pi}{\omega} - \frac{2}{\omega}\tan^{-1}\frac{\omega V}{g}$.

If $\frac{\omega V}{g} \to \infty$, $t_{contact} \to \pi/\omega$ (half a period)

If $\frac{\omega V}{g} \to 0$, $t_{contact} \to \frac{2\pi}{\omega} - \frac{2V}{g} = \frac{\pi}{\omega}\left(2 - \frac{2\omega V}{g}\right)$

(tends towards a ~~half~~ whole period).

Completely unseen

6

Math Marks

3.1

Comm

Q3

a/ i) In antiplane strain:

$\tau_{yz} = \mu \dfrac{\partial w}{\partial y} \to \sigma$ as $x^2+y^2 \to \infty$

2 $\Rightarrow w \sim \dfrac{\sigma}{\mu} y$ as $x^2+y^2 \to \infty$.

In anti-plane strain: $\tau = \begin{pmatrix} 0 & 0 & \tau_{xz} \\ 0 & 0 & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & 0 \end{pmatrix}$

$\underline{n} = (n_x, n_y, 0)^T$

$\Rightarrow \quad \tau \underline{n} = (\tau_{xz} n_x + \tau_{yz} n_y)\underline{k}$

so on stress-free boundary: $0 = \mu\left(n_x \dfrac{\partial w}{\partial x} + n_y \dfrac{\partial w}{\partial y}\right)$

$= \mu \dfrac{\partial w}{\partial n}$.

2 Hence $\dfrac{\partial w}{\partial n} = 0$ on $\underline{x} \in \partial\Omega$.

ii) Consider the transformation: $z = \frac{c}{2}(\zeta + \zeta^{-1})$.

On $|\zeta| = e^{\varepsilon}$, let $\zeta = e^{\varepsilon + i\theta}$ so that:

$z = \frac{c}{2}\left(e^{\varepsilon + i\theta} + e^{-\varepsilon - i\theta}\right)$

$\underset{x+iy}{\parallel} = \frac{c}{\cancel{2}}\left[\cosh\varepsilon \cdot \cos\theta + i \sinh\varepsilon \sin\theta\right]$.

$\Rightarrow \begin{array}{l} x = c\cosh\varepsilon \cdot \cos\theta \\ y = c \cdot \sinh\varepsilon \cdot \sin\theta \end{array} \Bigg] \Rightarrow \left(\dfrac{x}{c\cosh\varepsilon}\right)^2 + \left(\dfrac{y}{c\cdot\sinh\varepsilon}\right)^2 = 1$

1 This is the ellipse.

1 Check conformality: $\dfrac{dz}{d\zeta} = \frac{c}{2}(1 - \zeta^{-2}) \rightsquigarrow$ only not conformal at $\zeta = 0, \pm 1$ (inside $|\zeta| = e^{\varepsilon}$.)

1 Check outside of ellipse $\to$ outside of circle:

as $\zeta \to \infty$, $z \sim \frac{c}{2}\zeta \to \infty$.

B — Explanation on Problem Sheet.

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(iii) Inverse mapping: $\zeta = \frac{z}{c} + \sqrt{\frac{z^2}{c^2} - 1}$ 3-2

(so that outside $\to$ outside again).

Comments B

iv) $w \sim \frac{\sigma y}{\mu}$ as $|z| \to \infty$, $z \sim c\zeta/2 \Rightarrow y \sim \frac{c}{2} r\sin\theta$.

$\Rightarrow w \sim \frac{\sigma c}{2\mu} \cdot r\sin\theta$ as $r \to \infty$.

Want to solve $\nabla^2 w = 0$

with $w \sim \frac{\sigma c}{2\mu} r\sin\theta$ as $r \to \infty$

and $\frac{\partial w}{\partial r} = 0$ on $r = e^{\varepsilon}$. $\left(\frac{\partial w}{\partial n} = 0 \right)$.

Let $w = f \cdot \sin\theta$

$\Rightarrow f'' + \frac{f'}{r} - f/r^2 = 0 \Rightarrow f = \frac{\sigma c}{2\mu} r + A/r$

But $f'(e^{\varepsilon}) = 0 \Rightarrow f = \frac{\sigma c}{2\mu}\left(r + \frac{e^{2\varepsilon}}{r} \right)$.

4 1

$$\Rightarrow w = \frac{\sigma c}{2\mu}\left(r\sin\theta + \frac{e^{2\varepsilon}}{r}\sin\theta \right)$$

$$= \frac{\sigma c}{2\mu} \operatorname{Im}\left[\zeta - \frac{e^{2\varepsilon}}{\zeta} \right].$$

[But $\zeta = \frac{z}{c} + \sqrt{\frac{z^2}{c^2} - 1}$

and $\zeta^{-1} = \frac{2z}{c} - \zeta \Rightarrow = \frac{z}{c} - \sqrt{\frac{z^2}{c^2} - 1}$

$\Rightarrow w = \frac{\sigma c}{2\mu} \operatorname{Im}\left[\frac{z}{c} + \sqrt{\frac{z^2}{c^2} - 1} - e^{2\varepsilon}\left(\frac{z}{c} - \sqrt{\frac{z^2}{c^2} - 1} \right) \right]$

$= \frac{\sigma c}{2\mu} \operatorname{Im}\left[\frac{z}{c}(1 - e^{2\varepsilon}) + (1 + e^{2\varepsilon})\sqrt{\frac{z^2}{c^2} - 1} \right]$.

b) Now want

$$\left.\begin{aligned} \mu \frac{\partial w}{\partial x} &\sim \sigma \cdot \cos\alpha \\ \mu \frac{\partial w}{\partial y} &\sim \sigma \sin\alpha \end{aligned}\right] \text{ as } x^2+y^2 \to \infty .$$

$$\Rightarrow w \sim \frac{\sigma}{\mu}\cos\alpha \cdot x + \frac{\sigma}{\mu}\sin\alpha \cdot y$$

2 $\Rightarrow$ ~~$\frac{\partial w}{\partial r} = \cos\theta \frac{\partial w}{\partial x} + \sin\theta \frac{\partial w}{\partial y}$~~

$$\Rightarrow w \sim \frac{\sigma c}{2\mu}\left(r \sin\theta \cdot \sin\alpha + r\cos\theta \cdot \cos\alpha\right)$$

as $r \to \infty$

again $\frac{\partial w}{\partial r} = 0$ on $r = e^{\varepsilon}$.

2 Let $w = f(r)\cos\theta + g(r)\sin\theta$.

then $f = f_+ r + f_- / r = \frac{\sigma c}{2\mu}\sin\alpha\left(r + e^{2\varepsilon}/r\right)$

$g = g_+ r + g_- / r = \frac{\sigma c}{2\mu}\cos\alpha\left(r + e^{2\varepsilon}/r\right)$.

2

$$\therefore w = \frac{\sigma c}{2\mu}\left[\sin\alpha\left(r\sin\theta + \frac{e^{2\varepsilon}\sin\theta}{r}\right) + \cos\alpha\left(r\cos\theta + \frac{e^{2\varepsilon}\cos\theta}{r}\right)\right]$$

$$= \frac{\sigma c}{2\mu}\left[\sin\alpha \operatorname{Im}\left[\zeta - e^{2\varepsilon}/\zeta\right] + \cos\alpha\left[\zeta + \frac{e^{2\varepsilon}}{\zeta}\right]\right]$$

$$= \frac{\sigma c}{2\mu}\left\{\sin\alpha \operatorname{Im}\left[\frac{z}{c} + \sqrt{\frac{z^2}{c^2}-1} - e^{2\varepsilon}\left(\frac{z}{c} - \sqrt{\frac{z^2}{c^2}-1}\right)\right] + \cos\alpha \operatorname{Re}\left[\frac{z}{c} + \sqrt{\frac{z^2}{c^2}-1} + e^{2\varepsilon}\left(\frac{z}{c} - \sqrt{\frac{z^2}{c^2}-1}\right)\right]\right\}$$

2 as $\varepsilon \to 0$: $w \to \frac{\sigma c}{\mu}\left\{\sin\alpha \operatorname{Im}\sqrt{\frac{z^2}{c^2}-1} + \cos\alpha \operatorname{Re} z/c\right\}$.

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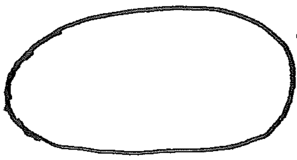
$\Rightarrow$ applying at angle reduces intensity of singularities at $z = \pm c$.

1

c). $\alpha = \pi/2$, $\varepsilon \ll 1$.

$$W = \frac{\sigma}{\mu} \operatorname{Im} \sqrt{\frac{z^2}{c^2} - 1}.$$



$$\frac{x^2}{c^2\cosh^2\varepsilon} + \frac{y^2}{c^2\sinh^2\varepsilon} = 1.$$

$$\Rightarrow x = c\cosh\varepsilon \sqrt{1 - \frac{y^2}{c^2\sinh^2\varepsilon}}.$$

$$0 = \frac{x \cdot x_y}{c^2\cosh^2\varepsilon} + \frac{y}{c^2\sinh^2\varepsilon}$$

$$\Rightarrow \quad x_y \simeq - \frac{y\cosh^2\varepsilon}{\pm c\sinh^2\varepsilon}.$$

$$x_y^2 + x \cdot x_{yy} = -\coth^2\varepsilon.$$

So close to the tip, $x_y = 0$ $(\because y \simeq 0)$
$|x| \simeq c$

2

$$\frac{1}{r_0} = |x_{yy}| \simeq \frac{\coth^2\varepsilon}{c} \sim \frac{1}{\varepsilon^2 c}.$$

as $\varepsilon \to 0$.

If $w = \operatorname{Im}[f(z)]$
then: $\tau_{yz} = \mu \operatorname{Re}[f'(z)]$

$$= \frac{\mu\sigma}{2\mu} \operatorname{Re}\left[(1 - e^{2\varepsilon}) + \frac{z}{\sqrt{z^2 - c^2}} \cdot (1 + e^{2\varepsilon})\right]$$

At crack tip $z \to c\cosh\varepsilon$

2

$$\Rightarrow \quad \tau_{yz} = \frac{\sigma}{2} \operatorname{Re}\left[(1 - e^{2\varepsilon}) + \frac{c \cdot \cosh\varepsilon}{c\sinh\varepsilon}(1 + e^{2\varepsilon})\right]$$

$$\simeq \frac{\sigma}{2}\left[\frac{2}{\varepsilon}\right] = \sigma/\varepsilon \sim \sigma\sqrt{c/r_0}$$ as desired

Curvature of tip not discussed in lectures. Only considered $\varepsilon = 0$, not $0 < \varepsilon \ll 1$.